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On Problems Concerning the Bruhat Decomposition and Structure Constants of Hecke Algebras of Finite Chevalley Groups

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This paper is dedicated to the memory of Donald G. Higman

1. Introduction

Let G be a finite Chevalley group over a finite field $k = F_q$ of characteristic p (as in [15] or [3]). Let B be a Borel subgroup of G with $U = O_p(B)$ (the unipotent radical of B), and let T be a maximal torus such that $B = UT$. Let $W = N_G(T)/T$ be the Weyl group of G. Then W is a finite Coxeter group with distinguished generators $S = \{s_1, \dots, s_n\}$.

Let Φ be the root system associated with W, with $\{\alpha_1, \dots, \alpha_n\}$ the set of simple roots corresponding to the generators $s_i \in S$, and let Φ_+ (resp. Φ_-) be the set of positive (resp. negative) roots associated with them.

By the Bruhat decomposition, the (U,U)-double cosets are parameterized by the elements of $N = N_G(T)$ and the (B,B)-double cosets are parameterized by the elements of W. The main result is a description of the set

$$B\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}$$

of representatives of the left B-cosets in the intersection

$$BwB \cap yBx^{-1}B,$$

for elements $\dot{w}, \dot{x}, \dot{y}$ in N corresponding to elements w, x, y in W, by an algorithm based on a reduced expression of w in terms of the generators $s_1, \dots, s_n$ of W. Its cardinality was shown by Iwahori [12] to be the structure constant

$$[e_w e_x : e_y]$$

for standard basis elements e_w, e_x, e_y of the Iwahori Hecke algebra $\mathcal{H}(G,B)$. A formula for the structure constants $[e_w e_x : e_y]$ was proved by Kawanaka [13] and is stated as follows:

$$[e_w e_x : e_y] = \sum_{\tau} q^{a(\tau)}(q-1)^{b(\tau)},$$

where the sum is taken over a set of subexpressions τ of a fixed reduced expression of w. The subexpressions are called *K-sequences for* (w,x,y) in what follows and were first defined in Kawanaka's paper [13]. The nonnegative integers $a(\tau)$ and

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