Michigan Math. J. 58 (2009)

Points and Hyperplanes of the Universal Embedding Space of the Dual Polar Space $DW(5,q)$, q Odd

B. N. Cooperstein & B. De Bruyn

Dedicated to the memory of Donald G. Higman

1. Introduction

A *partial linear rank-2 incidence geometry,* also called a *point-line geometry,* is a pair $\Gamma = (\mathcal{P}, \mathcal{L})$ consisting of a set $\mathcal{P}$ whose elements are called *points* and a collection $\mathcal{L}$ of distinguished subsets of $\mathcal{P}$ whose elements are called *lines,* such that any two distinct points are contained in at most one line. The *point-collinearity graph* of Γ is the graph with vertex set $\mathcal{P}$ where two points are adjacent if they are collinear (i.e., lie on a common line). By a *subspace* of Γ we mean a subset S of $\mathcal{P}$ such that, if $l \in \mathcal{L}$ and $l \cap S$ contains at least two points, then $l \subset S$. A subspace S is *singular* if each pair of points in S is collinear—that is, if S is a clique in the collinearity graph of Γ. We say that $(\mathcal{P}, \mathcal{L})$ is a *Gamma space* (see [13]) if, for every $x \in \mathcal{P}$, $\{x\} \cup \Gamma(x)$ is a subspace. A subspace $S \neq \mathcal{P}$ is a *geometric hyperplane* if it meets every line.

Let e be a positive integer, p a prime, and V a 6-dimensional vector space over the finite field $\mathbb{F}_q$, $q = p^e$, equipped with a nondegenerate alternating form f. Then every vector $\bar{u} \in V$ is *isotropic,* that is, satisfies $f(\bar{u}, \bar{u}) = 0$. A subspace U of V is called *totally isotropic* (*with respect to* f) if $f(\bar{u}_1, \bar{u}_2) = 0$ for all $\bar{u}_1, \bar{u}_2 \in U$.

Associated with (V, f) is a polar space denoted by $W(5,q)$. Here, by a *polar space* we mean a point-line geometry (P, L) that satisfies the following properties:

1. (P, L) is a Gamma space and, for every point p and line l, p is collinear with some point of l (this means that p is collinear with one point or all points of l);
2. no point p is collinear with every other point; and
3. there is an integer n called the *rank* of (P, L) such that, if $S_0 \subset S_1 \subset \cdots \subset S_k$ is a properly ascending chain of singular subspaces, then $k \leq n$.

When $n = 2$, (P, L) is said to be a *generalized quadrangle*.

The points (resp. lines) of $W(5,q)$ are the 1-dimensional (resp. 2-dimensional) subspaces of V that are totally isotropic with respect to f and where incidence is containment. In $W(5,q)$, two points $\langle \bar{u}_1 \rangle_V$ and $\langle \bar{u}_2 \rangle_V$ are collinear if and only if $f(\bar{u}_1, \bar{u}_2) = 0$ (i.e., iff $\bar{u}_1$ and $\bar{u}_2$ are orthogonal).

Received April 17, 2007. Revision received June 13, 2008.

The research of the second author is supported by a grant of the Fund for Scientific Research – Flanders (Belgium).