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Sobolev Peano Cubes

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In memoriam: Juha M. Heinonen (1960–2007)

1. Introduction

A classical theorem of Hahn [8] and Mazurkiewicz [19] states that X is a locally connected continuum if and only if there exists a continuous surjection $f\colon [0,1] \to X$. Since any cube $[0,1]^n$ is a continuous image of $[0,1]$, an equivalent statement is: X is a locally connected continuum if and only if there exists a continuous surjection $f\colon [0,1]^n \to X$.

The purpose of this paper is to generalize the Hahn–Mazurkiewicz theorem to differentiable and weakly differentiable mappings. Not surprisingly, our assumptions on X will be stronger.

Following Kirchheim [15], we say that a map $f\colon \Omega \to X$ from an open set $\Omega \subset \mathbb{R}^n$ to a metric space X is *metrically differentiable* at $x \in \Omega$ if there is a seminorm $\|\cdot\|_x$ on $\mathbb{R}^n$ such that

$$d(f(x), f(y)) - \|y-x\|_x = o(|y-x|) \quad \text{for } y \in \Omega. \tag{1.1}$$

The seminorm assumption means that $\|a+b\|_x \le \|a\|_x + \|b\|_x$ and $\|ta\|_x = |t|\|a\|_x$ but $\|\cdot\|_x$ can vanish on a linear subspace on $\mathbb{R}^n$, and (1.1) means that

$$\lim_{y\to x} \frac{d(f(x), f(y)) - \|y-x\|_x}{|y-x|} = 0.$$

Clearly, if $f\colon \Omega \to \mathbb{R}^k$ is (classically) differentiable at $x \in \Omega$, then it is metrically differentiable with $\|u\|_x = |Df(x)(u)|$. It is also easy to see that $f\colon (a,b) \to X$ is metrically differentiable at $x \in (a,b)$ if and only if the limit

$$\lim_{h\to 0} \frac{d(f(x+h), f(x))}{|h|}$$

exists and is finite.

A classical theorem of Rademacher [6] says that Lipschitz continuous functions $f\colon \Omega \to \mathbb{R}^k$ are differentiable a.e. Kirchheim [15] generalized this theorem to the case of metric space–valued mappings as follows.

Theorem 1.1 (Kirchheim). *A Lipschitz continuous map $f\colon \Omega \to X$ from an open set $\Omega \subset \mathbb{R}^n$ to a metric space X is metrically differentiable a.e.*

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