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Zeros of Regular Functions and Polynomials of a Quaternionic Variable

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1. Introduction

Let $\mathbb{H}$ denote the skew field of real quaternions. Its elements are of the form $q = x_0 + ix_1 + jx_2 + kx_3$, where the x_l are real and i, j, k are imaginary units (i.e., the square of each equals -1) such that $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$. The richness of the theory of holomorphic functions of one complex variable, along with motivations from physics, aroused a natural interest in a theory of quaternion-valued functions of a quaternionic variable. In fact, in the last century, several interesting theories have been introduced. The best known is due to Fueter [3; 4; 5], who defined the differential operator

$$\frac{\partial}{\partial \bar{q}} = \frac{1}{4}\left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\right),$$

now known as the Cauchy–Fueter operator, and defined the space of regular functions as the space of solutions of the equation associated to this operator. All regular functions are harmonic, and Fueter proved that this definition led to close analogues of Cauchy's theorem, Cauchy's integral formula, and the Laurent expansion. This theory is extremely successful and is now very well developed in many different directions. We refer the reader to [14] for the basic features of these functions. More recent work in this subject includes [1] and [9] and the references therein. Regarding the zero set of a Fueter-regular function, we remark here that it does not necessarily consist of isolated points and that its real dimension can be 0, 1, 2, or 4.

Inspired by an idea of Cullen [2], Gentili and Struppa [6; 7] have offered an alternative definition and theory of regularity for functions of a quaternionic variable. Cullen-regular functions are not harmonic in general. This new theory allows the study of natural power series (and polynomials) with quaternionic coefficients, which is excluded when the Fueter approach is followed. The papers [6] and [7] also include a study of the first properties of the zero set of Cullen-regular functions.

In order to present the definition of Cullen regularity, we start by using $\mathbb{S}$ to denote the 2-dimensional sphere of imaginary units of $\mathbb{H}$; that is, $\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}$. The definition given by Cullen can then be rephrased as follows.

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