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First-order Univalence Criteria, Interior Chord-arc Conditions, and Quasidisks

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Dedicated to Professor Christian Pommerenke on the occasion of his seventy-fifth birthday

1. Introduction

In all that follows, G and R will denote domains in the complex plane $\mathbb{C}$; G will always be simply connected, and $0 \notin R$. A first-order univalence criterion for G is a condition of the form

$$f'(G) \subset R, \tag{1.1}$$

or, somewhat more generally (and for present considerations more conveniently) of the form

$$\log f'(G) \subset R', \tag{1.2}$$

which implies that f is univalent on G. By (1.2) we mean of course that $f'(z) = e^{g(z)}$, where $g(G) \subset R'$. We will be concerned in large measure with the particular case in which $R' = \alpha S_0$, where S_0 is the infinite strip

$$S_0 = \{z : -1 < \Re\{z\} < 1\}.$$

The third author has studied the problem of determining criteria of the form (1.1) for smoothly bounded Jordan domains G that are sharp in the sense that there is no R_1 properly containing R for which the condition $f'(G) \subset R_1$ implies univalence (see [Ge] and the references therein). In this paper we examine two further aspects of first-order univalence criteria. First of all, in Section 2 we prove a theorem from which it follows immediately that there will be a criterion of either of these forms if and only if G satisfies an interior chord-arc condition—that is, if and only if there is an L such that

$$l(z_1, z_2) = \inf\left\{\int_\gamma |dz| : \gamma \subset G,\ z_1, z_2 \in \gamma\right\} \le L|z_1 - z_2| \tag{1.3}$$

for all $z_1, z_2 \in G$, where the γ are arcs.

To put our other results in context, we briefly discuss univalence criteria on quasidisks involving the pre-Schwarzian derivative $P_f(z) = f''(z)/f'(z)$ and the Schwarzian derivative $S_f(z) = (P_f(z))' - \frac{1}{2}(P_f(z))^2$. There is a classical result of Ahlfors [A] to the effect that, if G is a quasidisk, then there is some $\beta_S > 0$ such that

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