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The Möbius Geometry of Hypersurfaces

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1. Introduction

Suppose r is a defining function for a twice differentiable hypersurface $M^{2n-1} \subset \mathbb{C}^n$ near $p \in M$. In complex form, the Taylor expansion for r is given by

$$r(p+t) = r(p) + 2\,\text{Real}\sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p)t_j + L_{r,p}(t,\bar{t}) + \text{Real}\,Q_{r,p}(t,t) + o(|t|^2),$$

where $t = (t_1, \dots, t_n)$,

$$L_{r,p}(s,\bar{t}) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p)s_j\bar{t}_k,$$

and

$$Q_{r,p}(s,t) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial z_k}(p)s_j t_k.$$

It is a familiar fact in several complex variables that the hermitian quadratic form $L_{r,p}$ is invariant under biholomorphism. (Restricted to the complex tangent space, this is exactly the Levi form.) It is less familiar that the non-hermitian form $Q_{r,p}$ is invariant under Möbius transformation when restricted to the complex tangent space. This is established in Section 2.

Our main result is the following.

Theorem 1. *Suppose that $M^{2n-1} \subset \mathbb{C}^n$ is a non–Levi-flat, three times differentiable hypersurface and that, for all $p \in M$,*

$$Q_{r,p}(s,s) = 0 \quad \text{for } s = (s_1, \dots, s_n) \text{ with } \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p)s_j = 0. \tag{1}$$

Then M is contained in a hermitian quadric surface in $\mathbb{C}^n$.

Condition (1) is independent of the choice of defining function.

The proof of Theorem 1 uses the structural equations for a hypersurface and is similar to a proof the author used for an earlier characterization of the Bochner–Martinelli kernel [2]. An earlier analytic proof of Theorem 1 that requires the

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