Michigan Math. J. 56 (2008)

Special Loci in Moduli of Marked Curves

Cui Yin

1. Introduction

This paper concerns "special loci" in the moduli space $\mathfrak{M}_{g,[n]}$ parameterizing the smooth projective curves of genus g with n unordered marked points. Classically, one fixes a finite-order diffeomorphism φ of a compact orientable topological surface S of genus g with n marked points. The *special locus* associated to φ corresponds to the set of complex structures that can be put on S such that φ is an automorphism of the associated marked algebraic curve. The main theorem of this paper (Theorem 2.18) uses scheme theory to reformulate the notion of special locus in purely algebraic terms. A related result (Corollary 2.23) shows how the notion can often be further reformulated combinatorially in many cases. As a consequence of these results, the notion of special locus can be extended to curves over more general algebraically closed fields, including the characteristic-p case. In the last section, we consider some examples both in characteristic 0 and characteristic p. It turns out that special loci in characteristic p behave differently than the analogous special loci in characteristic 0 because of differences in the corresponding Riemann–Hurwitz formulas.

This paper builds upon previous work by González-Díez, Harvey, and Schneps. González-Díez and Harvey [GoH] considered curves of genus $g \geq 2$ over the complex numbers without marked points, and they studied the loci of those with a given automorphism group acting in a specified topological way. Cornalba [C] gave a complete classification of the irreducible subvarieties corresponding to the curves whose automorphism group contains a fixed cyclic subgroup of prime order in the case $g \geq 1$, $n = 0$, over the complex numbers (but without specifying the topological action). Later, Schneps [Sc1] considered the cyclic case for genus 0 with n marked points and for genus 1 with $n = 1$ or 2 marked points, which correspond to specifying a finite-order diffeomorphism of the underlying real 2-manifold. Related work has also been done by Magaard, Shaska, Shpectorov, and Völklein [MSSV], where the group but not the topological behavior is specified.

The main result in Section 2 provides a purely algebraic definition of special locus using scheme theory and without reference to differential or topological notions. In order to do this, we rely on the fact that the classical ("differential") special locus is irreducible. What we do is define two automorphisms α, α' of

Received December 29, 2006. Revision received September 8, 2008.