Michigan Math. J. 56 (2008)

A Minimal Brieskorn 5-Sphere in the Gromoll–Meyer Sphere and Its Applications

Carlos Durán & Thomas Püttmann

1. Introduction

This paper links two previously more or less unrelated important examples at the intersection of the fields of transformation groups, exotic spheres, and nonnegative curvature. These two examples are the exotic Gromoll–Meyer sphere Σ^7 and the Brieskorn sphere W_3^5. Several applications are drawn from the interplay between the Riemannian geometry of a 2-parameter family of metrics on Σ^7 and the equivariant geometry of W_3^5, which, surprisingly, determines the equivariant geometry of Σ^7 much more than the equivariant geometry of the exotic Brieskorn spheres $W^7_{6j-1,3}$, although the latter contain W_3^5 in a much more obvious way.

In 1974, Gromoll and Meyer [GrMy] constructed Σ^7 as a biquotient of the compact group Sp(2) and thereby the first exotic sphere with nonnegative sectional curvature. Note that Σ^7 can be regarded as the basic example of a biquotient in Riemannian geometry and, simultaneously, as the basic example of an exotic sphere. It generates the group $\Theta_7 \approx \mathbb{Z}_{28}$ of homotopy spheres in dimension 7, the first dimension except possibly 4 where exotic spheres can occur. Recently, it was shown that Σ^7 is actually the only exotic sphere that can be modeled by a biquotient of a compact Lie group [KaZ; To].

Because of this exceptional status of the Gromoll–Meyer sphere, it seems natural to study the geometry of Σ^7 in detail. Papers that do this from various viewpoints are [D; EK; PaSp; Y; Wi], for example. Here we investigate Σ^7 through the interaction between symmetry arguments, submanifold stratifications, and geodesic constructions. It is important, however, to note that we consider not only the Gromoll–Meyer metric on Σ^7 but also the entire 2-parameter family of metrics $\langle\cdot,\cdot\rangle_{\mu,\nu}$ that are $O(2)\times SO(3)$ invariant by construction. This family includes the Gromoll–Meyer metric $\left(\mu=\nu=\frac{1}{2}\right)$ and the pointed wiedersehen metric constructed in [D] ($\mu=\nu=1$) but not the metrics of almost positive sectional curvature obtained in [EK] and [Wi]. Extending the constructions of [D] and [ADPR], we obtain the following structural information.

Received June 20, 2007. Revision received December 19, 2007.

The first author was supported by FAPESP grant 03/016789 and FAEPEX grant 15406. The second author was supported by a DFG Heisenberg fellowship and by the DFG priority program SPP 1154 “Globale Differentialgeometrie”.