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Real Multiplication on K3 Surfaces and Kuga–Satake Varieties

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A K3-type Hodge structure is a simple, rational, polarized weight-2 Hodge structure V with $\dim V^{2,0} = 1$. Zarhin [Z] proved that the endomorphism algebra of a K3-type Hodge structure is either a totally real field or a CM field. Conversely, a K3-type Hodge structure whose endomorphism algebra is a given such field exists under fairly obvious conditions. For the totally real case, see Lemma 3.2.

In a manner similar to the case of abelian varieties and their polarized weight-1 Hodge structures, given a polarization and a totally positive endomorphism, one can define a new polarization (see Lemma 4.2). For a polarized abelian variety, this follows from the well-known relation between the Rosati invariant endomorphisms and the Néron–Severi group. In case the K3-type Hodge structure is a Hodge substructure of the H^2 of a smooth surface, it comes with a natural polarization induced by the cup product. It is then interesting to consider whether the polarization obtained by means of a totally real element a is also realized as the natural polarization for some other surface S_a. Thus $H^2(S_a)$ has a Hodge substructure isomorphic to the original one, but the isomorphism does not preserve the natural polarizations.

It follows easily from general results on K3 surfaces that, under a condition on the dimension of the Hodge structure, such K3 surfaces do exist (see Section 4.7). The isomorphism of Hodge substructures, in combination with the Hodge conjecture, then leads one to wonder whether there is an algebraic cycle realizing the isomorphism. We discuss some aspects of this question in Section 7. Mukai [Mu] proved that Hodge isometries between rational Hodge structures of K3 surfaces are realized by algebraic cycles, but this deep result does not apply to the general case. It does imply that if the endomorphism algebra of the (transcendental) Hodge structure of the K3 surface is a CM field, then any endomorphism is induced by an algebraic cycle on the self-product of the surface [Ma].

We consider the Kuga–Satake variety of a K3-type Hodge structure with real multiplication in Sections 5 and 6. The CM case was already studied in [vG2]. In particular, we consider the endomorphism algebra of the Kuga–Satake variety in the presence of real multiplications on the Hodge structure and we discuss some examples. We show that the Kuga–Satake construction is related to the corestriction of (Clifford) algebras. From this result we obtain a better understanding of previous work of Mumford [Mum] and Galluzzi [Ga].

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