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On the Inertia Group of Elliptic Curves in the Cremona Group of the Plane

Jérémy Blanc

1. Introduction

We work on some algebraically closed field $\mathbb{K}$. Let $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{K})$ be the projective plane over $\mathbb{K}$, let $\mathrm{Bir}(\mathbb{P}^2)$ be its group of birational transformations, and let $C \subset \mathbb{P}^2$ be an irreducible curve. The *decomposition group* of C in $\mathrm{Bir}(\mathbb{P}^2)$, introduced in [G], is the group

$$\mathrm{Dec}(C) = \mathrm{Bir}(\mathbb{P}^2)_C = \{g \in \mathrm{Bir}(\mathbb{P}^2) \mid g(C) \subset C,\ g|_C \colon C \dashrightarrow C \text{ is birational}\}.$$

The *inertia group* of C in $\mathrm{Bir}(\mathbb{P}^2)$, also introduced in [G], is the group

$$\mathrm{Ine}(C) = \mathrm{Bir}(\mathbb{P}^2)_{0C} = \{g \in \mathrm{Bir}(\mathbb{P}^2)_C \mid g(p) = p \text{ for a general point } p \in C\}.$$

(In our context, since the variety $\mathbb{P}^2$ and the inherent group $\mathrm{Bir}(\mathbb{P}^2)$ will not change, we will prefer the notation $\mathrm{Dec}(C)$ and $\mathrm{Ine}(C)$ to that of Gizatullin.)

If φ is a birational transformation of $\mathbb{P}^2$ that does not collapse C (this latter condition is always true if C is nonrational), then φ conjugates the group $\mathrm{Dec}(C)$ (resp. $\mathrm{Ine}(C)$) to the group $\mathrm{Dec}(\varphi(C))$ (resp. $\mathrm{Ine}(\varphi(C))$). The conjugacy classes of the two groups are thus birational invariants.

On one hand, these groups are useful for describing the birational equivalence of curves of the plane. On the other hand, given two groups, the curves fixed by the elements are useful for deciding whether the groups are birationally conjugate and, moreover, are often the unique invariant needed (see [BaB; BBl; Bl; F]).

In the case where $\mathbb{K} = \mathbb{C}$, the inertia groups of curves of geometric genus ≥ 2 have been classically studied (see [C]); a modern precise classification may be found in [BlPV]. For the case of the decomposition groups, we refer to [P1; P2] and the references therein.

In this article we will study the case of the inertia group of curves of geometric genus 1 and, in particular, the case of plane smooth cubic curves, which constitute the only case in which nontrivial elements are known. We state now the three main results that we prove.

First is a Noether–Castelnuovo-like theorem for the generators of the inertia group. (The same result holds for the decomposition group; see [P2, Thm. 1.4].)

Theorem 1. *The inertia group of a smooth plane cubic curve is generated by its elements of degree* 3, *which are—except the identity—its elements of lower degree*.

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