Michigan Math. J. 56 (2008)

Axiomatic Regularity on Metric Spaces

Sergey A. Timoshin

Introduction

The problem of the regularity of solutions to partial differential equations with prescribed boundary values and of regular variational problems constitutes one of the most interesting chapters in analysis, which has its origins mostly starting from the year 1900, when Hilbert formulated his famous 23 problems in an address delivered before the International Congress of Mathematicians at Paris. The essential parts of the 20th problem on existence of solutions and its related 19th problem about the regularity itself read as follows.

19th problem: "Are the solutions of regular problems in the calculus of variations always necessarily analytic?"

20th problem: "Has not every regular variational problem a solution, provided certain assumptions regarding the given boundary conditions are satisfied, and provided also if need be that the notion of a solution shall be suitably extended?"

It is known that in the Euclidean space the problem of minimizing a variational integral in a set of functions with prescribed boundary values is closely related to solving the corresponding Dirichlet problem for its Euler–Lagrange equation. In particular, for the Dirichlet p-energy integral

$$\int_{\Omega\subset\mathbb{R}^n} |\nabla u(x)|^p\, dx,$$

the corresponding Euler–Lagrange equation, for $1 < p < \infty$, is

$$\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = 0.$$

Starting with the remarkable result of Bernstein in 1904 that any C^3 solution of an elliptic nonlinear analytic equation in two variables is necessarily analytic, and through the works of many authors, in particular in the works of Leray and Schauder in 1934, it was proved that every sufficiently smooth, say $C^{0,\alpha}$ (Hölder continuous), stationary point of a regular variational problem with analytic integrand is analytic. On the other hand, by direct methods of the calculus of variations one can prove in general the existence of solutions that have derivatives only in a generalized sense and satisfy the equation only in a correspondingly weak form.

Received January 4, 2007. Revision received March 27, 2008.