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# Corona Theorem for $H^\infty$ on Coverings of Riemann Surfaces of Finite Type

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## 1. Introduction

*1.1*

Let $X$ be a complex manifold and let $H^\infty(X)$ be the Banach algebra of bounded holomorphic functions on $X$ equipped with the supremum norm. We assume that $X$ is Carathéodory hyperbolic; that is, the functions in $H^\infty(X)$ separate the points of $X$. The maximal ideal space $\mathcal{M} = \mathcal{M}(H^\infty(X))$ is the set of all nonzero linear multiplicative functionals on $H^\infty(X)$. Since the norm of each $\phi \in \mathcal{M}$ is $\le 1$, $\mathcal{M}$ is a subset of the closed unit ball of the dual space $(H^\infty(X))^*$. It is a compact Hausdorff space in the Gelfand topology (i.e., in the weak-$*$ topology induced by $(H^\infty(X))^*$). Further, there is a continuous embedding $i: X \hookrightarrow \mathcal{M}$ taking $x \in X$ to the evaluation homomorphism $f \mapsto f(x)$, $f \in H^\infty(X)$. The complement to the closure of $i(X)$ in $\mathcal{M}$ is called the *corona*. The *corona problem* is: Given $X$, determine whether the corona is empty. For example, according to Carleson's celebrated corona theorem [C], this is true for $X$ the open unit disk in $\mathbb{C}$. (This was conjectured by Kakutani in 1941.) Also, there are nonplanar Riemann surfaces for which the corona is nontrivial (see e.g. [JM; Ga; BD; L] and references therein). The general problem for planar domains is still open, as is the problem in several variables for the ball and polydisk. (In fact, there are no known examples of domains in $\mathbb{C}^n$, $n \ge 2$, without corona.) At present, the strongest corona theorem for planar domains is due to Garnett and Jones [GJ]. It states that the corona is empty for any Denjoy domain—that is, a domain of the form $\bar{\mathbb{C}} \setminus E$ where $E \subset \mathbb{R}$.

The corona problem has the following analytic reformulation (see e.g. [G]): A collection $f_1, \dots, f_n$ of functions from $H^\infty(X)$ satisfies the *corona condition* if

$$1 \ge \max_{1 \le j \le n} |f_j(x)| \ge \delta > 0 \quad \text{for all } x \in X. \tag{1.1}$$

The corona problem being solvable (i.e., the corona is empty) means that the Bezout equation

$$f_1 g_1 + \cdots + f_n g_n \equiv 1 \tag{1.2}$$

has a solution $g_1, \dots, g_n \in H^\infty(X)$ for any $f_1, \dots, f_n$ satisfying the corona condition. We refer to $\max_{1 \le j \le n} \|g_j\|_\infty$ as a "bound on the corona solutions". (Here $\|\cdot\|_\infty$ is the norm on $H^\infty(X)$.)

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