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# Mass Flow for Noncompact Manifolds

Ricardo Berlanga

## 1. Introduction

The group of homeomorphisms acts on the space of measures on a manifold $M$ in such a way that, for each measure $\mu$ on $M$ and each homeomorphism $h\colon M \to M$, the action $h_*\mu$ of $h$ on $\mu$ is defined by $h_*\mu(E) = \mu(h^{-1}(E))$ for each Borel set $E \subseteq M$. The group of homeomorphisms preserving a given measure is just the stabilizer of that measure under the action. In 1941, Oxtoby and Ulam [12] characterized the orbit of standard Lebesgue measure on the unit cube under this action. Since then, Oxtoby and Ulam's result has been of enormous importance for the study of groups of measure-preserving homeomorphisms.

The aim of our work is to generalize and reformulate previous research of Fathi [9] on the definition and properties of the so-called mass flow homomorphism to the $\sigma$-compact case. Our "noncompact" methods allow us to replace Fathi's use of handles and tubular neighborhoods by neighborhoods alone. Therefore, we do not depend on the existence of a combinatorial structure on the base manifold to proceed with the argumentation. As a by-product, simplification of Fathi's arguments is gained.

Let $M$ be a connected, second-countable manifold without boundary equipped with a "good" measure $\mu_o$ (see Section 2). Let $\mathcal{H}_c(M, \mu_o)$ be the group of $\mu_o$-measure-preserving homeomorphisms of $M$ with compact support endowed with the Whitney topology, and let $\mathcal{H}_{c,o}(M, \mu_o)$ be the path component of the identity. The *mass flow homomorphism* is a group homomorphism from the universal covering space of $\mathcal{H}_{c,o}(M, \mu_o)$, thought of as a space of paths modulo homotopy (rel. $\partial I$), to the first homology group $H_1(M, \mathbb{R})$.

Fathi's approach to the mass flow, in which homology is viewed as a set of homotopy classes of maps into the circle, is suitable only for compact manifolds. In order to extend Fathi's theory to noncompact manifolds we define, in Section 4, a new version of the mass flow homomorphism that relies on a real homology theory based on measures due to W. Thurston. In Section 5, both approaches are compared (see Proposition 5.4 and the subsequent comment). Via a quotient process, in Section 6 the mass flow homomorphism on $\mathcal{H}_{c,o}(M, \mu_o)$ is effectively defined, giving rise to an important commutative diagram that is studied in some detail.

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