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The Direct Sum Decomposability of ^eM in Dimension 2

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Dedicated to Professor Melvin Hochster on the occasion of his sixty-fifth birthday

0. Introduction

Unless explicitly stated otherwise, throughout this paper we assume that R is a Noetherian ring of prime characteristic p and that M is a finitely generated R-module. By $(R, \mathfrak{m}, k)$ we indicate that R is local with its maximal ideal $\mathfrak{m}$ and its residue field $k = R/\mathfrak{m}$. We always denote $q := p^e$ for varying $e \in \mathbb{N}$.

For every $e \in \mathbb{N}$, there exists a Frobenius map (which is a ring homomorphism) $F^e: R \to R$ defined by $F^e(r) = r^q = r^{p^e}$ for any $r \in R$. Thus, given M, there is a derived R-module structure, denoted by ^eM, on the same abelian group M but with its scalar multiplication determined by $r \cdot x = r^q x = r^{p^e} x$ for $r \in R$ and $x \in M$. It is routine to verify that $\operatorname{Ann}(M) \subseteq \operatorname{Ann}(^eM) \subseteq \sqrt{\operatorname{Ann}(M)}$ and that $\operatorname{Ass}(M) = \operatorname{Ass}(^eM)$ for all $e \in \mathbb{N}$.

When R is reduced it is clear that ^eR and $R^{1/q} := \{r^{1/p^e} \mid r \in R\}$ are isomorphic as R-modules for every e. Using this terminology, a result of Kunz [K1, Thm. 2.1] states that R is regular if and only if ^eR is flat over R for some $e \geq 1$ or, equivalently, for all $e \in \mathbb{N}$.

We say that R is *F-finite* if ^1R is finitely generated over R or, equivalently, if ^eR is finitely generated over R for all $e \in \mathbb{N}$. By a result of Kunz [K2], every F-finite ring is excellent. If R is F-finite and if M is a finitely generated R-module, then it is easy to see that ^eM remains finitely generated over R for every $e \in \mathbb{N}$.

Similarly, if ^1M is finitely generated over R then so is $^1(R/\operatorname{Ann}(M))$. This means that $R/\operatorname{Ann}(M)$ is an F-finite ring. In other words, $^e(R/\operatorname{Ann}(M))$ is finite over $R/\operatorname{Ann}(M)$ (or, equivalently, over R) for all e, which forces ^eM to be finitely generated over R for all $e \in \mathbb{N}$.

For any $e \in \mathbb{N}$, the derived R-module ^eM can be roughly identified as the module structure of M over the subring $R^q := \{r^q = r^{p^e} \mid r \in R\}$. Thus, in general, the "size" of ^eM should increase as $e \to \infty$. Assuming that ^eM is finite over R for all $e \in \mathbb{N}$, we are interested in whether it is possible for the derived R-modules ^eM to remain indecomposable (i.e., not writable as a direct sum of two nontrivial submodules) for all $e \in \mathbb{N}$. Since we can always replace R by $R/\operatorname{Ann}(M)$, we may simply assume that R is F-finite.

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