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The Chern Coefficients of Local Rings

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Dedicated to Professor Melvin Hochster on the occasion of his 65th birthday

1. Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$, and let I be an $\mathfrak{m}$-primary ideal. One of our goals is to study the set of I-good filtrations of R. More concretely, we will consider the set of multiplicative, decreasing filtrations of R ideals, $\mathcal{A} = \{I_n, I_0 = R, I_{n+1} = II_n, n \gg 0\}$, that is integral over the I-adic filtration and conveniently coded in the corresponding Rees algebra and its associated graded ring:

$$\mathcal{R}(\mathcal{A}) = \sum_{n\geq 0} I_n t^n, \qquad \operatorname{gr}_{\mathcal{A}}(R) = \sum_{n\geq 0} I_n/I_{n+1}.$$

In this paper we study certain strata of these algebras. For that we will focus on the role of the Hilbert polynomial of the Hilbert function $\lambda(R/I_{n+1})$, $n \gg 0$,

$$H^1_{\mathcal{A}}(n) = P^1_{\mathcal{A}}(n) = \sum_{i=0}^{d}(-1)^i e_i(\mathcal{A})\binom{n+d-i}{d-i},$$

particularly of its coefficients $e_0(\mathcal{A})$ and $e_1(\mathcal{A})$. Two of our principal aims are to establish relationships between the coefficients $e_i(\mathcal{A})$ for $i = 0, 1$ and (marginally) $e_2(\mathcal{A})$. If R is a Cohen–Macaulay ring then there are numerous related developments; see especially those given and discussed in [8] and [28]. The situation is quite different in the non–Cohen–Macaulay case. Just to illustrate the issue, suppose $d > 2$ and consider a comparison between $e_0(I)$ and $e_1(I)$, a subject that has received considerable attention. It is often possible to pass to a reduction $R \to S$, with $\dim S = 2$ or even $\dim S = 1$, so that $e_0(I) = e_0(IS)$ and $e_1(I) = e_1(IS)$. If R is Cohen–Macaulay, then this is straightforward. However, in general the relationship between $e_0(IS)$ and $e_1(IS)$ may involve other invariants of S, some of which may not be easily traceable all the way to R.

Our perspective is partly influenced by the interpretation of the coefficient e_1 as a *tracking number* (see [6] for the original terminology and also [26])—that is, as a numerical positional tag of the algebra $\mathcal{R}(\mathcal{A})$ in the set of all such algebras with the same multiplicity. The coefficient e_1 under various circumstances is also called the *Chern number* or *Chern coefficient* of the algebra.

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