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Generalized Test Ideals and Symbolic Powers

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Dedicated to Professor Mel Hochster
on the occasion of his sixty-fifth birthday

Introduction

Ein, Lazarsfeld, and Smith proved in [ELS] the following uniform behavior of symbolic powers of ideals in affine regular rings of equal characteristic 0: If h is the largest height of any associate prime of an ideal $I \subseteq R$, then $I^{(hn+kn)} \subseteq (I^{(k+1)})^n$ for all integers $n \geq 1$ and $k \geq 0$. Here, if W is the complement of the union of the associate primes of I, then the mth symbolic power $I^{(m)}$ of I is defined to be the contraction of $I^m R_W$ to R, where R_W is the localization of R at the multiplicative system W. To prove this, the authors introduced the notion of asymptotic multiplier ideals, which is a variant of multiplier ideals associated to filtrations of ideals and is formulated in terms of resolution of singularities. The uniform behavior of symbolic powers follows immediately from a combination of properties (of asymptotic multiplier ideals) whose proofs require deep vanishing theorems.

In [HHu5], Hochster and Huneke generalized the result in [ELS] to the case of arbitrary regular rings of equal characteristic (i.e., both of equal characteristic 0 and of positive prime characteristic) in a completely different way. Furthermore, they used in [HHu6] similar ideas to prove more subtle behaviors of symbolic powers of ideals in a regular ring of equal characteristic. Their methods depend on the theory of tight closure and reduction to positive characteristic and thus require neither resolution of singularities nor vanishing theorems, which are proved only in characteristic 0. In this paper, by combining the ideas of Ein–Lazarsfeld–Smith and Hochster–Huneke, we give a slight generalization of Hochster and Huneke's results in [HHu6].

Tight closure is an operation defined on ideals or modules in positive characteristic; it was introduced by Hochster and Huneke [HHu2] in the 1980s. The test ideal $\tau(R)$ of a Noetherian ring R of prime characteristic p is the annihilator ideal of all tight closure relations in R, and it plays a central role in the theory of tight closure. In [HaY] and [Ha], Hara and Yoshida introduced a generalization of the test ideal $\tau(R)$, the ideal $\tau(\mathfrak{a}_\bullet)$ associated to a filtration of ideals $\mathfrak{a}_\bullet$, and

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