Michigan Math. J. 57 (2008)

Toric Geometry of Cuts and Splits

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1. Introduction

With any finite graph $G = (V, E)$ we associate a projective toric variety X_G over a field $\mathbb{K}$ as follows. The coordinates $q_{A|B}$ of the ambient projective space are indexed by unordered partitions $A|B$ of the vertex set V. The dense torus has two coordinates (s_{ij}, t_{ij}) for each edge $\{i, j\} \in E$. The polynomial rings in these two sets of unknowns are

$$\mathbb{K}[q] := \mathbb{K}[q_{A|B} \mid A \cup B = V,\ A \cap B = \emptyset],$$
$$\mathbb{K}[s,t] := \mathbb{K}[s_{ij}, t_{ij} \mid \{i, j\} \in E].$$

Each partition $A|B$ of the vertex set V defines a subset $\mathrm{Cut}(A|B)$ of the edge set E. Namely, $\mathrm{Cut}(A|B)$ is the set of edges $\{i, j\}$ such that $i \in A$, $j \in B$ or $i \in B$, $j \in A$. The variety we wish to study is specified by the following homomorphism of polynomial rings:

$$\phi_G : \mathbb{K}[q] \to \mathbb{K}[s,t], \qquad q_{A|B} \mapsto \prod_{\{i,j\}\in \mathrm{Cut}(A|B)} s_{ij} \cdot \prod_{\{i,j\}\in E\setminus \mathrm{Cut}(A|B)} t_{ij}. \tag{1.1}$$

One may wish to think of s and t as abbreviations for "separated" and "together". The kernel of ϕ_G is a homogeneous toric ideal I_G, which we call the *cut ideal* of the graph G. We are interested in the projective toric variety X_G that is defined by the cut ideal I_G.

EXAMPLE 1.1. Let $G = K_4$ be the complete graph on four nodes, so $V = \{1, 2, 3, 4\}$ and $E = \{12, 13, 14, 23, 24, 34\}$. The ring map ϕ_{K_4} is specified by

$$\begin{aligned}
q_{|1234} &\mapsto t_{12}t_{13}t_{14}t_{23}t_{24}t_{34}, & q_{1|234} &\mapsto s_{12}s_{13}s_{14}t_{23}t_{24}t_{34},\\
q_{12|34} &\mapsto t_{12}s_{13}s_{14}s_{23}s_{24}t_{34}, & q_{2|134} &\mapsto s_{12}t_{13}t_{14}s_{23}s_{24}t_{34},\\
q_{13|24} &\mapsto s_{12}t_{13}s_{14}s_{23}t_{24}s_{34}, & q_{3|124} &\mapsto t_{12}s_{13}t_{14}s_{23}t_{24}s_{34},\\
q_{14|23} &\mapsto s_{12}s_{13}t_{14}t_{23}s_{24}s_{34}, & q_{4|123} &\mapsto t_{12}t_{13}s_{14}t_{23}s_{24}s_{34}.
\end{aligned}$$

The cut ideal for the complete graph on four nodes is the principal ideal

$$I_{K_4} = \langle q_{|1234}q_{12|34}q_{13|24}q_{14|23} - q_{1|234}q_{2|134}q_{3|124}q_{123|4}\rangle.$$

Thus the toric variety X_{K_4} defined by I_{K_4} is a quartic hypersurface in $\mathbb{P}^7$.

Received July 3, 2006. Revision received February 12, 2007.
B. Sturmfels was partially supported by the National Science Foundation (DMS-0456960).