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Longest Alternating Subsequences of Permutations

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Dedicated to Mel Hochster on the occasion of his sixty-fifth birthday

1. Introduction

Let $\mathfrak{S}_n$ denote the symmetric group of permutations of $1, 2, \dots, n$, and let $w = w_1 \cdots w_n \in \mathfrak{S}_n$. An *increasing subsequence* of w of length k is a subsequence $w_{i_1} \cdots w_{i_k}$ satisfying

$$w_{i_1} < w_{i_2} < \cdots < w_{i_k}.$$

There has been much recent work on the length $\text{is}_n(w)$ of the longest increasing subsequence of a permutation $w \in \mathfrak{S}_n$. A highlight is the asymptotic determination of the expectation $E(n)$ of is_n by Logan–Shepp [11] and Vershik–Kerov [18]:

$$E(n) := \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \text{is}_n(w) \sim 2\sqrt{n}, \quad n \to \infty. \tag{1}$$

Baik, Deift, and Johansson [3] obtained a vast strengthening of this result—namely, the limiting distribution of $\text{is}_n(w)$ as $n \to \infty$. In particular, for w chosen uniformly from $\mathfrak{S}_n$,

$$\lim_{n \to \infty} \text{Prob}\left(\frac{\text{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \le t \right) = F(t), \tag{2}$$

where $F(t)$ is the Tracy–Widom distribution. The proof uses a result of Gessel [9] that gives a generating function for the quantity

$$u_k(n) = \#\{w \in \mathfrak{S}_n : \text{is}(w) \le k\}.$$

Namely, define

$$U_k(x) = \sum_{n \ge 0} u_k(n) \frac{x^{2n}}{n!^2}, \quad k \ge 1;$$

$$I_i(2x) = \sum_{n \ge 0} \frac{x^{2n+i}}{n!\,(n+i)!}, \quad i \in \mathbb{Z}.$$

The function I_i is the *hyperbolic Bessel function* of the first kind of order i. Note that $I_i(2x) = I_{-i}(2x)$. Gessel then showed that

$$U_k(x) = \det(I_{i-j}(2x))_{i,j=1}^{k}.$$

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