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Noncommutative Resolutions and Rational Singularities

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Dedicated to Mel Hochster on the occasion of his 65th birthday

1. Introduction

Throughout the paper, k will denote a fixed algebraically closed field of characteristic 0 and, unless otherwise specified, all rings will be k-algebras. Our aim is to show that the center of a homologically homogeneous, finitely generated k-algebra has rational singularities; in particular, if a finitely generated normal commutative k-algebra has a noncommutative crepant resolution (as introduced by the second author), then it has rational singularities.

We begin by setting this result in context and defining the relevant terms. Suppose that $X = \operatorname{Spec} R$ for an affine (i.e., finitely generated) normal Gorenstein k-algebra R. The nicest form of resolution of singularities $f: Y \to X$ occurs when f is *crepant* in the sense that $f^*\omega_X = \omega_Y$. Even when they exist, crepant resolutions need not be unique, but they are related—indeed, Bondal and Orlov conjectured in [BoO2] (see also [BoO1]) that two such resolutions should be derived equivalent.

Bridgeland [Bri] proved the Bondal–Orlov conjecture in dimension 3. The second author observed in [V3] that Bridgeland's proof could be explained in terms of a *third* crepant resolution of X that is now noncommutative (the definition will be given in what follows). This and similar observations by others have led to a number of different approaches to the Bondal–Orlov conjecture and related topics (see e.g. [Be; BeKa; Ch; IR; Ka2; Kaw]).

It is therefore natural to ask how the existence of a noncommutative crepant resolution affects the original commutative singularity. It is well known, and follows easily from [KoMo, Thm. 5.10], that if a Gorenstein singularity has a crepant resolution then it has rational singularities. So it is logical to ask, as raised in [V2, Ques. 3.2], *is this true for a noncommutative crepant resolution?* Here we answer this question affirmatively.

Let Δ be a prime affine k-algebra that is finitely generated as a module over its center $Z(\Delta)$. Mimicking [BH], we say that Δ is *homologically homogeneous of*

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