Michigan Math. J. 57 (2008)

Rational Singularities Associated to Pairs

Karl Schwede & Shunsuke Takagi

Dedicated to Professor Mel Hochster
on the occasion of his sixty-fifth birthday

1. Introduction and Background

Rational singularities are a class of singularities that have been heavily studied since their introduction in the 1960s. Roughly speaking, an algebraic variety has rational singularities if its structure sheaf has the same cohomology as the structure sheaf of a resolution of singularities. Rational singularities enjoy many useful properties; in particular, they are both normal and Cohen–Macaulay. Furthermore, many common varieties have rational singularities, including toric varieties and quotient varieties. Rational singularities are also known to be closely related to the singularities of the minimal model program. In particular, it is known that log terminal singularities are rational and that Gorenstein rational singularities are canonical.

There is, however, an important distinction between rational singularities and singularities of the minimal model program. In the minimal model program it is natural to consider pairs (X, D), where X is a variety and D is a $\mathbb{Q}$-divisor. In recent years the study of pairs $(X, \mathfrak{a}^c)$, where $\mathfrak{a}$ is an ideal sheaf and c is a positive real number, has also become quite common. Thus it is natural to try to extend the notion of rational singularities to pairs. We define two notions of rational pairs: a *rational pair,* which is analogous to a Kawamata log terminal (klt) pair; and a *purely rational triple,* which is analogous to a purely log terminal (plt) triple (we will discuss the characteristic-p analogues in Section 5). It is hoped that these definitions and their study will help further the understanding both of rational singularities and of log terminal pairs.

In characteristic 0, defining rational singularities for pairs has one distinct advantage over the corresponding variants of log terminal singularities. In order for (X, D) to be log terminal, one necessarily must have $K_X + D$ a $\mathbb{Q}$-Cartier divisor. Likewise, for the pair $(X, \mathfrak{a}^c)$ to be log terminal, X must necessarily be $\mathbb{Q}$-Gorenstein. One can define rational singularities for a pair $(X, \mathfrak{a}^c)$ without any such conditions on X.

Received August 14, 2007. Revision received January 17, 2008.

The first author was partially supported by RTG grant number 0502170 and also as a National Science Foundation post-doc. The second author was partially supported by Grant-in-Aid for Young Scientists (B) 17740021 from JSPS and by Program for Improvement of Research Environment for Young Researchers from SCF commissioned by MEXT of Japan.