

Almost Regular Sequences and the Monomial Conjecture

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To Mel Hochster on his 65th birthday

The monomial conjecture of Mel Hochster has been one of the most important open problems in commutative algebra for many years. The conjecture is as follows.

CONJECTURE. *Let R be a Noetherian local ring of dimension d , and suppose that x_1, x_2, \dots, x_d is a system of parameters for R . Then for all integers $t \geq 0$ we have*

$$x_1^t x_2^t \cdots x_d^t \notin (x_1^{t+1}, \dots, x_d^{t+1}),$$

where $(x_1^{t+1}, \dots, x_d^{t+1})$ denotes the ideal generated by $x_1^{t+1}, \dots, x_d^{t+1}$.

This conjecture has assumed a central role because it is simply stated and it implies several other important conjectures, notably the canonical element conjecture, for rings of positive or mixed characteristic. In fact, when this conjecture was first announced it had numerous further consequences, some of which (such as the new intersection conjecture) were later proved by different means. We refer to Hochster [6; 8] for descriptions of these conjectures and their status at various times.

The monomial conjecture is almost trivial for rings that contain the rational numbers and is not difficult for rings of positive characteristic, but it remains an open problem for rings of mixed characteristic. The most recent advance was made by Heitmann [5], who proved it in mixed characteristic in dimension 3.

One traditional method for approaching this and other conjectures has been to construct Cohen–Macaulay modules for which a system of parameters for the ring becomes a regular sequence. It is unknown whether one can find finitely generated modules with this property, but Hochster showed many years ago that for equicharacteristic rings one can find infinitely generated modules (and even algebras) with this property (see [6]).

In the course of Heitmann’s proof, he shows that a weaker condition than being a regular sequence suffices to prove these conjectures. We call a sequence of elements with this property an *almost regular sequence* and we give a precise definition in Section 1.

In this paper we first review some of the known facts about almost regular sequences and then discuss some related questions in the equicharacteristic case. Finally, we discuss a variation on this concept for rings of mixed characteristic and its relation to the monomial conjecture.

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