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Rationality of Hilbert–Kunz Multiplicities: A Likely Counterexample

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1. A Conjecture

At a 2004 Banff workshop, I gave a talk to demonstrate that, in many cases of interest, the Hilbert–Kunz multiplicity of a hypersurface is a rational number. (Mel Hochster, in the audience, told me a curious general fact: the set of possible Hilbert–Kunz multiplicities is countable.)

At the time I suspected that Hilbert–Kunz multiplicities must be rational. But soon after the workshop I found reason to change my opinion, and in this paper I suggest that a certain hypersurface defined by a 5-variable polynomial has $\frac{4}{3} + \frac{5}{14\sqrt{7}}$ as its Hilbert–Kunz multiplicity.

Throughout, q will denote a power 2^n of 2 with $n \geq 0$, and H will be the element $x^3 + y^3 + xyz$ of $\mathbb{Z}/2[x, y, z]$; $e_n(H^j)$ is the colength, $\deg(x^q, y^q, z^q, H^j)$, of the ideal (x^q, y^q, z^q, H^j). It is known [1, Thm. 3] that $e_n(H)$ is $\frac{7q^2-q-3}{3}$ or $\frac{7q^2-q-5}{3}$ according as $q \equiv 1$ or 2 modulo 3. I'll present conjectured formulas of similar type for $e_n(H^j)$, with j arbitrary, that are strongly supported by computer calculation. I show that if these hold then the Hilbert–Kunz multiplicity of $uv + H(x, y, z)$ is $\frac{4}{3} + \frac{5}{14\sqrt{7}}$.

Explicitly, I define numbers u_j and v_j and conjecture that, if $q \geq j$, then $e_n(H^j) = \frac{jq(7q-j)}{3} + u_j$ or $\frac{jq(7q-j)}{3} + v_j$ according as $q \equiv 1$ or 2 modulo 3. The definition of u_j and v_j is complicated and may appear to be unmotivated. In fact, it is related to ideas from [2], and the reader will find a somewhat less mysterious form of our conjecture, connected to these ideas, in Section 3 of this paper.

To define u_j and v_j, I introduce some notation.

Definition 1.1. Γ is the free abelian group on symbols $[0], [1], [2], \dots$ and E. σ_0 and σ_1 are the endomorphisms of Γ that satisfy the following statements.

(1) $\sigma_0([i]) = [i+1]$ for even i and $[i-1] + E$ for odd i; $\sigma_0(E) = 2E$.

(2) $\sigma_1([i]) = [i-1] + E$ for even $i \neq 0$ and $[i+1]$ for odd i; $\sigma_1([0]) = [0]$ and $\sigma_1(E) = 2E$.

Definition 1.2. If $0 \leq j < q$ then we define an element $f(q, j)$ of Γ inductively as follows:

$$f(1,0) = [0], \quad f(2q, 2k) = \sigma_0 f(q,k), \quad f(2q, 2k+1) = \sigma_1 f(q,k).$$

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