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A Property of the Absolute Integral Closure of an Excellent Local Domain in Mixed Characteristic

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Dedicated to Professor Melvin Hochster on the occasion of his sixty-fifth birthday

1. Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local excellent domain and let R^+ be the absolute integral closure of R—that is, the integral closure of R in the algebraic closure of the fraction field of R. The ring R^+ when R is 3-dimensional and of mixed characteristic played an important role in Heitmann's proof of the direct summand conjecture in dimension 3 [3]. In dimension > 3 the direct summand conjecture is still open. This motivates the study of R^+ in mixed characteristic and in dimension > 3.

Hochster and Huneke [4] proved that if R contains a field of characteristic 0 then R^+ is a big Cohen–Macaulay R-algebra; in other words, $H^i_{\mathfrak{m}}(R^+) = 0$ for all $i < \dim R$, and every system of parameters of R is a regular sequence on R^+. Recently, in joint work with Huneke [5], we gave a simpler proof of this result.

This paper is motivated by Huneke's suggestion that perhaps the techniques of our paper [5] could be applied to R^+ in mixed characteristic. Our main result is the following theorem.

Theorem 1.1. *Let $(R, \mathfrak{m})$ be a Noetherian local excellent domain of mixed characteristic, residual characteristic $p > 0$, and dimension ≥ 3. Let $\sqrt{pR}$ (resp. $\sqrt{pR^+}$) be the radical of the principal ideal of R (resp. R^+) generated by p. Set $\bar{R} = R/\sqrt{pR}$ (resp. $\overline{R^+} = R^+/\sqrt{pR^+}$). Then*

(i) *$H^1_{\mathfrak{m}}(\overline{R^+}) = 0$, and*
(ii) *every part of a system of parameters $\{a, b\}$ of $\bar{R}$ of length 2 is a regular sequence on $\overline{R^+}$.*

This theorem suggests the following.

Question. Let $(R, \mathfrak{m})$ be a Noetherian local excellent domain of mixed characteristic. Is $\overline{R^+}$ then a big Cohen–Macaulay $\bar{R}$-algebra? That is:

(i) is $H^i_{\mathfrak{m}}(\overline{R^+}) = 0$ for all $i < \dim \bar{R}$; and
(ii) is every system of parameters of $\bar{R}$ a regular sequence on $\overline{R^+}$?

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