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A Vanishing Theorem for Finitely Supported Ideals in Regular Local Rings

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To Mel Hochster, on the occasion of his 65th birthday

Introduction

In [L5, p. 747, (b)] there is a vanishing conjecture for an ideal I in a d-dimensional regular local ring $(R,\mathfrak{m})$. (A stronger "CM" conjecture on that page was disproved by Hyry [Hy, p. 389, Ex. 3.6].) Suppose there is a map $f: X \to \operatorname{Spec}(R)$ that factors as a finite sequence of blowups with smooth centers and is such that $I\mathcal{O}_X$ is invertible. Let E be the closed fiber $f^{-1}\{\mathfrak{m}\}$. The conjecture is that

$$\mathrm{H}^i_E(X,(I\mathcal{O}_X)^{-1}) = 0 \quad \text{for all } i \neq d.$$

This statement implies, with $\ell(I)$ denoting the analytic spread of I and $\widetilde{\ }$ denoting "adjoint ideal of" (a.k.a. multiplier ideal with exponent 1), that

$$\widetilde{I^{n+1}} = I\widetilde{I^n} \quad \text{for all } n \geq \ell(I)-1,$$

which in turn implies a number of "Briançon-Skoda with coefficients" results; see [L5, pp. 745–746]. The conjectured statement holds true when $d=2$, and it was proved by Cutkosky [Cu] for R essentially of finite type over a field of characteristic 0 (in which case it is closely related to vanishing theorems in the theory of multiplier ideals; see [La]). In these two situations, the assumed principalization f is known to exist for any $I \neq (0)$.

In this paper we show that vanishing holds for those R-ideals that are *finitely supported*—in other words, those for which there is a sequence of blowups (as before) in which all the centers are closed points.

In addition, we deduce that the adjoint ideal of a finitely supported ideal I is itself finitely supported, with point basis obtained by subtracting $\min(d-1, r_\beta)$ componentwise from the point basis (r_β) of I. (The terminology is explained in Sec. 3.)

More consequences of vanishing are scattered throughout Sections 3–4. For example, for finitely supported I, Proposition 3.4 generalizes the $\widetilde{I^{n+1}} = I\widetilde{I^n}$ relation; furthermore, if I is the integral closure $\bar{J}$ of a d-generated ideal J—whence $J\widetilde{I^{d-1}} = \widetilde{I^d}$—then Proposition 4.2 gives that $J\widetilde{I^{d-2}} = \widetilde{I^{d-1}} \cap J \neq \widetilde{I^{d-1}}$ (unless

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