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Oriented Cohomology, Borel–Moore Homology, and Algebraic Cobordism

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Warmly dedicated to Mel Hochster, who, among other important lessons, taught me that sometimes my pencil is smarter than my brain

Introduction

The notion of *oriented cohomology* has been introduced, in various forms and in various settings, in the work of Panin [10], Levine and Morel [6], and others. A related notion, that of *oriented Borel–Moore homology,* appears in [6]. Mocanasu [7] has examined the relation of these two notions and, with a somewhat different axiomatic than appears in either [6] or [10], has given an equivalence of the two theories; the relation is that the cohomology with supports in a closed subset X of a smooth scheme M becomes the Borel–Moore homology of X.

Our main goal in this paper is to tie all these theories together. Our first step is to extend results of [10] in order to show that an orientation on a ring cohomology theory gives rise to a good theory of projective push-forwards on the cohomology with supports. This extension of Panin's results allows us to use the ideas and results of Mocanasu, which in essence show that many of the properties and structures associated with the cohomology of a smooth scheme M with supports in a closed subset X depend only on X; we require resolution of singularities for this step. We axiomatize this into the notion of an *oriented duality theory,* which one can view as a version of the classical notion of a Bloch–Ogus twisted duality theory. The main difference between a general oriented duality theory (H, A) and a Bloch–Ogus theory is that we do not assume that the Chern class map $L \mapsto c_1(L)$ satisfies the usual additivity with respect to tensor product of line bundles:

$$c_1(L \otimes M) = c_1(L) + c_1(M).$$

This relation is replaced by the *formal group law* $F_A(u, v) \in A(\operatorname{Spec} k)[[u, v]]$ of the underlying oriented cohomology theory A, defined by the relation

$$c_1(L \otimes M) = F_A(c_1(L), c_1(M)).$$

In fact, the Chern classes $c_1(L)$ and formal group law F_A are not explicitly given as part of the axioms but instead follow from the more basic structures—namely, the pull-back, the projective push-forward, and the projective bundle formula.

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