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Syzygies of Multiplier Ideals on Singular Varieties

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Dedicated to Mel Hochster on the occasion of his sixty-fifth birthday

Introduction

It was recently established in [10] that multiplier ideals on a smooth variety satisfy some special syzygetic properties. The purpose of this paper is to show how some of these can be extended to the singular setting.

To set the stage, we review some of the results from [10]. Let X be a smooth complex variety of dimension $\dim(X) = d$, and denote by $(\mathcal{O}, \mathfrak{m})$ the local ring of X at a fixed point $x \in X$. Let $\mathcal{J} \subseteq \mathcal{O}$ be any multiplier ideal; that is, assume $\mathcal{J}$ is the stalk at x of a multiplier ideal sheaf $\mathcal{J}(X, \mathfrak{b}^{\lambda})$, where $\mathfrak{b} \subseteq \mathcal{O}_X$ is an ideal sheaf and λ is a positive rational number. The main result of [10] is that if $p \geq 1$ then no minimal pth syzygy of $\mathcal{J}$ vanishes modulo $\mathfrak{m}^{d+1-p}$ at x. In other words, if we consider a minimal free resolution of the ideal $\mathcal{J}$ over the regular local ring $\mathcal{O}$,

$$\cdots \xrightarrow{u_3} F_2 \xrightarrow{u_2} F_1 \xrightarrow{u_1} F_0 \longrightarrow \mathcal{J} \longrightarrow 0,$$

then no minimal generator of the pth syzygy module

$$\operatorname{Syz}_p(\mathcal{J}) \stackrel{\text{def}}{=} \operatorname{Im}(u_p) \subseteq F_{p-1}$$

of $\mathcal{J}$ lies in $\mathfrak{m}^{d+1-p} \cdot F_{p-1}$. Although this result places no restriction on the orders of vanishing of the *generators* of $\mathcal{J}$, it provides strong constraints on the first and higher *syzygies* of $\mathcal{J}$. When $d = 2$ these conditions hold for any integrally closed ideal, but [10] shows that in dimensions $d \geq 3$ only rather special integrally closed ideals can arise as multiplier ideals. (In contrast, it was established by Favre–Jonsson [2] and Lipman–Watanabe [12] that any integrally closed ideal on a smooth *surface* is locally a multiplier ideal.)

Multiplier ideals can be defined on any **Q**-Gorenstein variety X or, more generally, for any pair (X, Δ) consisting of an effective Weil **Q**-divisor Δ on a normal variety X such that $K_X + \Delta$ is **Q**-Cartier. It is natural to wonder whether multiplier ideals in this context satisfy the same sort of algebraic properties as in the smooth case. We will see (Example 3.1) that the result from [10] just quoted does not extend without change. However, we show that at least for *first* syzygies, one obtains a statement by replacing the maximal ideal $\mathfrak{m}$ by any parameter ideal.

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