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# Frobenius Splitting of Certain Rings of Invariants

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*Dedicated to Professor Melvin Hochster on the occasion of his sixty-fifth birthday*

## 1. Introduction

The concept of $F$-purity was introduced by Hochster and Roberts [6]; the $F$-purity for a Noetherian ring of prime characteristic is equivalent to the splitting of the Frobenius map when the ring is finitely generated over its subring of $p$th powers. It is closely related to the Frobenius splitting property à la Mehta and Ramanathan [11] for algebraic varieties; more precisely, the $F$-split property for an irreducible projective variety $X$ over an algebraically closed field of positive characteristic is equivalent to the $F$-purity of the ring $\bigoplus_{n\geq 0} H^0(X; L^n)$ for any ample line bundle $L$ over $X$ (cf. [3; 13; 14]). We feel that it is only appropriate to dedicate this paper to Professor Hochster on the occasion of his sixty-fifth birthday and thus make a modest contribution to this birthday volume.

Let $k$ be an algebraically closed field of characteristic $p > 0$ and let $X$ be a $k$-scheme. One has the Frobenius morphism (which is only an $\mathbb{F}_p$-morphism) $F\colon X \to X$ defined as the identity map of the underlying topological space of $X$, where the morphism of structure sheaves $F^{\#}\colon \mathcal{O}_X \to \mathcal{O}_X$ is the $p$th power map. The morphism $F$ induces a morphism of $\mathcal{O}_X$-modules $\mathcal{O}_X \to F_*\mathcal{O}_X$. The variety $X$ is called *Frobenius split* (or *F-split,* or simply *split*) if there exists a splitting $\varphi\colon F_*\mathcal{O}_X \to \mathcal{O}_X$ of the morphism $\mathcal{O}_X \to F_*\mathcal{O}_X$. Equivalently, $X$ is Frobenius split if there exists a morphism of sheaves of abelian groups $\varphi\colon \mathcal{O}_X \to \mathcal{O}_X$ such that (i) $\varphi(f^p g) = f\varphi(g)$ with $f, g \in \mathcal{O}_X$ and (ii) $\varphi(1) = 1$. Basic examples of varieties that are Frobenius split are smooth affine varieties, toric varieties (cf. [1]), generalized flag varieties, and Schubert varieties [11]. Smooth projective curves of genus $> 1$ are examples of varieties that are *not* Frobenius split.

Frobenius splitting is an interesting property to study. If $X$ is Frobenius split, then it is weakly normal [1, Prop. 1.2.5] and reduced [1, Prop. 1.2.1]. Indeed, projective varieties that are Frobenius split are very special. We refer the reader to [1] for further details.

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