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A Local Ring such that the Map between Grothendieck Groups with Rational Coefficients Induced by Completion Is Not Injective

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Dedicated to Professor Melvin Hochster on the occasion of his 65th birthday

1. Introduction

In this paper, we construct a local ring A such that the kernel of the map $G_0(A)_{\mathbb{Q}} \to G_0(\hat{A})_{\mathbb{Q}}$ is not zero, where $\hat{A}$ is the completion of A with respect to the maximal ideal and where $G_0(\cdot)_{\mathbb{Q}}$ is the Grothendieck group of finitely generated modules with rational coefficients. In our example, A is a 2-dimensional local ring that is essentially of finite type over $\mathbb{C}$, but it is not normal.

For a Noetherian ring R, we set

$$G_0(R) = \frac{\bigoplus_{M:\ \text{f.g. } R\text{-mod.}} \mathbb{Z}[M]}{\langle [L] + [N] - [M] \mid 0 \to L \to M \to N \to 0 \text{ is exact}\rangle};$$

this is called the *Grothendieck group* of finitely generated R-modules. Here $[M]$ denotes the free basis corresponding to a finitely generated R-module (f.g. R-mod.) M of the free module $\bigoplus \mathbb{Z}[M]$, where $\mathbb{Z}$ is the ring of integers.

For a flat ring homomorphism $R \to A$, we have the induced map $G_0(R) \to G_0(A)$ defined by $[M] \mapsto [M \otimes_R A]$.

We are interested in the following problem (Question 1.4 in [7]).

Problem 1.1. *Let R be a Noetherian local ring. Is the map $G_0(R)_{\mathbb{Q}} \to G_0(\hat{R})_{\mathbb{Q}}$ injective?*

Here $\hat{R}$ denotes the $\mathfrak{m}$-adic completion of R, where $\mathfrak{m}$ is the unique maximal ideal of R. For an abelian group N, $N_{\mathbb{Q}}$ denotes the tensor product with the field of rational numbers $\mathbb{Q}$.

Next we explain motivation and applications.

Assume that S is a regular scheme and that X is a scheme of finite type over S. Then, by the singular Riemann–Roch theorem [3], we obtain an isomorphism

$$\tau_{X/S}\colon G_0(X)_{\mathbb{Q}} \xrightarrow{\sim} A_*(X)_{\mathbb{Q}},$$

where $G_0(X)$ (resp. $A_*(X)$) is the *Grothendieck group* of coherent sheaves on X (resp. *Chow group* of X). We refer the reader to Chapters 1, 18, and 20 in [3] for

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