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# F-Thresholds, Tight Closure, Integral Closure, and Multiplicity Bounds

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*Dedicated to Professor Mel Hochster on the occasion of his sixty-fifth birthday*

## Introduction

Let $R$ be a Noetherian ring of positive characteristic $p$. For every ideal $\mathfrak{a}$ in $R$, and for every ideal $J$ whose radical contains $\mathfrak{a}$, one can define asymptotic invariants that measure the containment of the powers of $\mathfrak{a}$ in the Frobenius powers of $J$. These invariants were introduced in the case of a regular local F-finite ring in [MTW], where it was shown that they coincide with the jumping exponents for the generalized test ideals of Hara and Yoshida [HaY]. In this paper we work in a general setting and show that the F-thresholds still capture interesting and subtle information. In particular, we relate them to tight closure and integral closure and also to multiplicities.

Given $\mathfrak{a}$ and $J$ as just described, we define for every positive integer $e$

$$\nu_{\mathfrak{a}}^{J}(p^e) := \max\{r \mid \mathfrak{a}^r \not\subseteq J^{[p^e]}\},$$

where $J^{[q]}$ is the ideal generated by the $p^e$-powers of the elements of $J$. We put

$$\mathrm{c}_{+}^{J}(\mathfrak{a}) := \limsup_{e\to\infty} \frac{\nu_{\mathfrak{a}}^{J}(p^e)}{p^e} \quad \text{and} \quad \mathrm{c}_{-}^{J}(\mathfrak{a}) := \liminf_{e\to\infty} \frac{\nu_{\mathfrak{a}}^{J}(p^e)}{p^e},$$

and if these two limits coincide then we denote their common value by $\mathrm{c}^{J}(\mathfrak{a})$ and call it the *F-threshold of* $\mathfrak{a}$ *with respect to* $J$.

Our first application of this notion is to the description of the tight closure and of the integral closure of parameter ideals. Suppose that $(R,\mathfrak{m})$ is a $d$-dimensional Noetherian local ring of positive characteristic and that $J$ is an ideal in $R$ generated by a full system of parameters. We show that, under mild conditions, for every ideal $I \supseteq J$ we have $I \subseteq J^*$ if and only if $\mathrm{c}_{+}^{I}(J) = d$ (and in this case $\mathrm{c}_{-}^{I}(J) = d$, too). We similarly show that, under suitable mild hypotheses, if $I \supseteq J$, then

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