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Adjoints of Ideals

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Adjoint ideals and multiplier ideals have recently emerged as a fundamental tool in commutative algebra and algebraic geometry. In characteristic 0 they may be defined using resolution of singularities. In positive prime characteristic p, Hara and Yoshida [4] introduced the analogue of multiplier ideals as generalized test ideals for a tight closure theory. For all characteristics, even mixed, Lipman gave the following definition.

Definition 0.1. Let R be a regular domain and I an ideal in R. Then the *adjoint* $\operatorname{adj} I$ of I is defined as

$$\operatorname{adj} I = \bigcap_v \{r \in R \mid v(r) \geq v(I) - v(J_{R_v/R})\},$$

where the intersection varies over all valuations v on the field of fractions K of R that are nonnegative on R and for which the corresponding valuation ring R_v is a localization of a finitely generated R-algebra. Here $J_{R_v/R}$ denotes the Jacobian ideal of R_v over R.

By our assumption on v, each valuation in the definition of $\operatorname{adj} I$ is Noetherian.

Many valuations v have the same valuation ring R_v; any two such valuations are positive real multiples of each other and are called *equivalent*. In Definition 0.1 we need only use one v from each equivalence class. In the sequel, we will always choose *normalized* valuations—that is, the integer-valued valuation v such that, for all $r \in R$, $v(r)$ equals the nonnegative integer n satisfying that rR_v equals the nth power of the maximal ideal of R_v.

Lipman proved that, for any ideal I in R and any $x \in R$, $\operatorname{adj}(xI) = x \operatorname{adj}(I)$. In particular, $\operatorname{adj}(xR) = (x)$.

A crucial and powerful property is the subadditivity of adjoints: $\operatorname{adj}(IJ) \subseteq \operatorname{adj}(I) \operatorname{adj}(J)$. This was proved in characteristic 0 by Demailly, Ein, and Lazarsfeld [3] and for generalized test ideals in characteristic p by Hara and Yoshida [4, Thm. 6.10]. A simpler proof in characteristic p can be found in [1, Lemma 2.10]. A version of subadditivity formula on singular varieties was proved by Takagi in [21]. But subadditivity of adjoints is unknown in general. We prove it for generalized monomial ideals in Section 4 and for ideals in two-dimensional regular

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