Michigan Math. J. 57 (2008)

Lifting Seminormality

Raymond C. Heitmann

Suppose R is a local Noetherian ring and y is a regular element contained in the maximal ideal of R. If R satisfies some nice property $(\star)$ then R/yR frequently does not satisfy $(\star)$, although there are exceptions—for example, when $(\star)$ is the Cohen–Macaulay property. On the other hand, many theorems state that $(\star)$ can be lifted from R/yR to R. If R/yR is an integral domain, respectfully reduced, then so is R. If $(\star)$ is regularity, the result is trivial. If $(\star)$ is normality, the result is well known and easy to prove; we will include a proof here simply to illustrate the relative levels of difficulty of this and our main result. However, when David Jaffe asked what happened when $(\star)$ was seminormality, a quick answer was not forthcoming. The purpose of this article is to show that seminormality can be lifted.

We should remark that the requirement for R to be a local Noetherian ring is important for this result and virtually all results of this type. There are non-Noetherian rings with a single maximal principal ideal yR and all kinds of pathological behavior, and the fact that R/yR is a field yields little. Likewise, if R has more than one maximal ideal, then passing to R/yR can "improve" R by removing maximal ideals P from the prime spectrum when R_P fails to satisfy $(\star)$.

Throughout this article, all rings are commutative with unity. Local rings are always Noetherian. The total quotient ring of R will be denoted by $Q(R)$, and the integral closure of R in $Q(R)$ will be denoted by R'. We will primarily be concerned with Noetherian rings, but excellence is not assumed and so R' need not be Noetherian. We begin with a quick proof of the well-known result that normality lifts. Here we consider only the domain case, but allowing R/yR to be reduced merely makes the proof slightly longer; the ideas in the proof remain the same. The same is true of the proof of our main theorem: restricting to the domain case does not make the problem any easier.

Theorem. *If R is a local integral domain, yR is a prime ideal in R, and R/yR is normal, then R is normal.*

Proof. We will show R to be normal by showing that it satisfies the Serre conditions (R1) and (S2). Suppose P is a height-1 prime ideal of R. If $P = yR$, then P principal implies R_P regular. If $P \neq yR$, then there exists a height-2 prime ideal Q of R that contains P and yR. Since R/yR satisfies (R1), it follows that

Received April 26, 2007. Revision received August 10, 2007.
The author was partially supported by NSF Grant no. DMS 0355486.