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Multi-Ideal-adic Completions of Noetherian Rings

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1. Introduction

Let R be a commutative ring with identity. A *filtration* on R is a decreasing sequence $\{I_n\}_{n=0}^{\infty}$ of ideals of R. Associated to a filtration is a well-defined completion $R^* = \varprojlim_n R/I_n$ and a canonical homomorphism $\psi\colon R \to R^*$ [13, Chap. 9]. If $\bigcap_{n=0}^{\infty} I_n = (0)$, then ψ is injective and R may be regarded as a subring of R^* [13, p. 401]. In the terminology of Northcott, a filtration $\{I_n\}_{n=0}^{\infty}$ is *multiplicative* if $I_0 = R$ and $I_n I_m \subseteq I_{n+m}$ for all $m \geq 0$ and $n \geq 0$ [13, p. 408]. A well-known example of a multiplicative filtration on R is the I-adic filtration $\{I^n\}_{n=0}^{\infty}$, where I is a fixed ideal of R.

In this paper we consider filtrations of ideals of R that are *not* multiplicative and examine the completions associated to these filtrations. We assume the ring R is Noetherian. Instead of successive powers of a fixed ideal I, we use a filtration formed from a more general descending sequence $\{I_n\}_{n=0}^{\infty}$ of ideals. We require, for each $n > 0$, that the nth ideal I_n be contained in the nth power of the Jacobson radical of R and that $I_{nk} \subseteq I_n^k$ for all $k, n \geq 0$. We call the associated completion a *multi-adic* completion and denote it by R^*. The basics of the multi-adic construction and the relationship between this completion and certain ideal-adic completions are considered in Section 2. In Section 3 we prove, for $\{I_n\}$ as just described, that R^* is Noetherian. Let $(R, \mathbf{m})$ be a Noetherian local ring. If R is excellent, Henselian, or universally catenary, we prove in Section 4 that R^* has the same property.

We were inspired to pursue this project partly because of our continuing interest in exploring completions and power series. In our previous work we constructed various examples of rings inside relatively well-understood rings such as the (x)-adic completion $k[y][[x]]$ of a polynomial ring $k[x, y]$ in two variables x and y over a field k [4; 5]. The examples we obtained demonstrate that certain properties of a ring may fail to extend to its $\mathbf{m}$-adic completion, where $\mathbf{m}$ is a maximal ideal [6].

The process of passing to completion gives an analytic flavor to algebra. Often we view completions in terms of power series or in terms of coherent sequences, as

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