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Local Cohomology on Diagrams of Schemes

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Dedicated to Professor Melvin Hochster
on the occasion of his sixty-fifth birthday

1. Introduction

Let S be a scheme, G a flat S-group scheme, and X a G-scheme (i.e., an S-scheme on which G acts). In [18], a G-linearization of an invertible sheaf on X is defined. As quasi-coherent sheaves are important in studying a scheme, G-linearized quasi-coherent sheaves are important in studying a scheme with a group action. If S, G, and $X = \operatorname{Spec} A$ are all affine, then the category $\operatorname{Lin}(G, X)$ of G-linearized quasi-coherent sheaves on X is equivalent to the category of (G, A)-modules (see [8]). In particular, if $S = \operatorname{Spec} k = X$ with k a field, then $\operatorname{Lin}(G, X)$ is equivalent to the category of G-modules. However, the definition of a G-linearization in [18] is complicated, and probably it is difficult to study the homological algebra of $\operatorname{Lin}(G, X)$ only from the definition. In [9], the diagram $B_G^M(X)$ of schemes is defined and the category of quasi-coherent sheaves $\operatorname{Qch}(G, X) = \operatorname{Qch}(B_G^M(X))$ is studied. Note that $\operatorname{Lin}(G, X)$ and $\operatorname{Qch}(G, X)$ are equivalent. The category $\operatorname{Qch}(X)$ of quasi-coherent sheaves on X is embedded in the category of $\mathcal{O}_X$-modules $\operatorname{Mod}(X)$, and this embedding gives some flexibility to the homological algebra of $\operatorname{Qch}(X)$. Similarly, $\operatorname{Qch}(G, X)$ is embedded in $\operatorname{Mod}(G, X) := \operatorname{Mod}(B_G^M(X))$, and the homological algebra of $\operatorname{Qch}(G, X)$ is considered in $\operatorname{Mod}(G, X)$. Note that $B_G^M(X)$ is a diagram of schemes of the form

$$G \times_S G \times_S X \;\overset{1_G \times a}{\underset{p_{23}}{\overset{\mu \times 1_X}{\rightrightarrows\!\!\!\rightarrow}}}\; G \times_S X \;\overset{a}{\underset{p_2}{\rightrightarrows}}\; X,$$

where $a\colon G \times_S X \to X$ is the action, $\mu\colon G \times_S G \to G$ is the product, and p_2 and p_{23} are appropriate projections. Thus, in the study of sheaves on diagrams of schemes, it is important to consider $\operatorname{Lin}(G, X)$.

Local cohomology is a powerful tool in commutative ring theory. The local cohomology $H_{\mathfrak{m}}^i$ on a local ring $(A, \mathfrak{m})$ is especially important. However, when we consider a group action, “local phenomena” sometimes occur on nonaffine schemes; see Example 8.19. Thus, to construct a theory of equivariant local

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