Michigan Math. J. 57 (2008)

Hilbert Functions over Toric Rings

Vesselin Gasharov, Noam Horwitz,
& Irena Peeva

1. Introduction

Throughout this paper, S stands for the polynomial ring $k[x_1, \dots, x_n]$ over a field k. The ring S is graded by $\deg(x_i) = 1$ for each i. The vector space of all polynomials of degree i is denoted by S_i. If J is a graded ideal, then J_i is the vector space of all polynomials in J of degree i. The Hilbert function

$$\begin{aligned} h\colon \mathbf{N} &\to \mathbf{N}, \\ i &\mapsto \dim_k J_i, \end{aligned}$$

is an important numerical invariant that measures the size of J. Macaulay's theorem [Ma] characterizes the Hilbert functions of homogeneous ideals in S. Macaulay's key idea is that every Hilbert function is attained by a lex ideal. Lex ideals are special monomial ideals defined in a simple combinatorial way. Macaulay's theorem was generalized to Betti numbers [Bi; Hu; Pa]: every lex ideal attains maximal Betti numbers among all homogenous ideals with the same Hilbert function. Furthermore, lex ideals play a key role in Hartshorne's proof of his famous result that the Hilbert scheme is connected [Ha]. These are important results, so it is interesting to find analogues over nonpolynomial rings. A lot of attention was given to the Clements–Lindström ring, which has the form $C = S/(x_1^{c_1}, \dots, x_n^{c_n})$ with $c_1 \le \dots \le c_n \le \infty$. Macaulay's theorem is known to hold in this case. Recently, there has been a lot of work on the lex-plus-powers conjecture. Another open conjecture [GHiP] is that every lex ideal in C attains maximal Betti numbers over C among all homogenous ideals in C with the same Hilbert function. The special case $c_1 = \dots = c_n = 2$ is well studied, and we have the following results.

Theorem 1.1. *Let $E = S/(x_1^2, \dots, x_n^2)$ (or one can assume that E is an exterior algebra). Then the following statements hold.*

(1) *For every graded ideal J in E, there exists a lex ideal with the same Hilbert function* [K; Kr].
(2) *The Hilbert scheme that parameterizes all graded ideals in E with a fixed Hilbert function h is connected. More precisely, every graded ideal in E with Hilbert function h is connected to the lex ideal with Hilbert function h* [PS1].

Received January 24, 2007. Revision received November 30, 2007.