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Ascent of Module Structures, Vanishing of Ext, and Extended Modules

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This paper is dedicated to Melvin Hochster on the occasion of his sixty-fifth birthday

Introduction

Suppose $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are commutative Noetherian local rings and $\varphi\colon R \to S$ is a flat local homomorphism with the property that the induced homomorphism $R/\mathfrak{m} \to S/\mathfrak{m}S$ is bijective. We consider natural questions of ascent and descent of modules between R and S: (i) Given a finitely generated R-module M, when does M have an S-module structure that is compatible with the R-module structure via φ? (ii) Given a finitely generated S-module N, is there a finitely generated R-module M such that N is S-isomorphic to $S \otimes_R M$ or (iii) S-isomorphic to a direct summand of $S \otimes_R M$?

In Section 1 we make some general observations about homomorphisms $R \to S$ satisfying the condition $R/\mathfrak{m} = S/\mathfrak{m}S$. We show that if a compatible S-module structure exists, then it arises in an obvious way: The natural map $M \to S \otimes_R M$ is an isomorphism. (One example to keep in mind is that of a finite-length module M when $S = \hat{R}$, the $\mathfrak{m}$-adic completion.) Moreover, if $R \to S$ is flat, then M has a compatible S-module structure if and only if $S \otimes_R M$ is finitely generated as an R-module.

In Section 2 we prove, assuming that $R \to S$ is flat, that M has a compatible S-module structure if and only if $\operatorname{Ext}^i_R(S, M)$ is finitely generated as an R-module for $i = 1, \dots, \dim_R(M)$. We were motivated to investigate this implication because of the following result of Buchweitz and Flenner [BF] and Frankild and Sather-Wagstaff [FS-W2]: A finitely generated R-module M is $\mathfrak{m}$-adically complete if and only if $\operatorname{Ext}^i_R(\hat{R}, M) = 0$ for all $i \geq 1$. Theorem 2.5 summarizes the main results of the first two sections. Note that it subsumes the result of [BF; FS-W2], but our proof here is quite different.

In Section 3 we address questions (ii) and (iii) and show that (iii) always has an affirmative answer when S is the Henselization, but not necessarily when S is the $\mathfrak{m}$-adic completion.

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