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The Core of Ideals in Arbitrary Characteristic

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Dedicated to Mel Hochster on the occasion of his sixty-fifth birthday

1. Introduction

In this paper we provide explicit formulas for the core of an ideal. Recall that for an ideal I in a Noetherian ring R, the *core* of I, $\operatorname{core}(I)$, is the intersection of all reductions of I. For a subideal $J \subset I$ we say that J is a *reduction* of I, or that I is *integral* over J, if $I^{r+1} = JI^r$ for some $r \geq 0$; the smallest such r is called the *reduction number* of I with respect to J and is denoted by $r_J(I)$. If $(R, \mathfrak{m})$ is local with infinite residue field k then every ideal has a *minimal reduction,* which is a reduction minimal with respect to inclusion. Minimal reductions of a given ideal I are far from unique, but they all share the same minimal number of generators, called the *analytic spread* of I and written $\ell(I)$. Minimal reductions arise from Noether normalizations of the *special fiber ring* $\mathcal{F}(I) = \operatorname{gr}_I(R) \otimes k$ of I, and therefore $\ell(I) = \dim \mathcal{F}(I)$. From this one readily sees that $\operatorname{ht} I \leq \ell(I) \leq \dim R$; these inequalities are equalities for any $\mathfrak{m}$-primary ideal, and if the first inequality is an equality then I is called *equimultiple*. Obviously, the core can be obtained as an intersection of minimal reductions of a given ideal.

Through the study of the core one hopes to better understand properties shared by all reductions. The notion was introduced by Rees and Sally for the purpose of generalizing the Briançon–Skoda Theorem [17]. As an a priori infinite intersection of reductions, the core is difficult to compute, and there have been considerable efforts to find explicit formulas; see [3; 4; 9; 10; 11; 12; 15]. We quote the following result from [15].

Theorem 1.1. *Let R be a local Gorenstein ring with infinite residue field k, let I be an R-ideal with $g = \operatorname{ht} I > 0$ and $\ell = \ell(I)$, and let J be a minimal reduction of I with $r = r_J(I)$. Assume that I satisfies G_ℓ, that* $\operatorname{depth} R/I^j \geq \dim R/I - j + 1$ *for* $1 \leq j \leq \ell - g$, *and that either* $\operatorname{char} k = 0$ *or* $\operatorname{char} k > r - \ell + g$. *Then*

$$\operatorname{core}(I) = J^{n+1} : I^n$$

for every $n \geq \max\{r - \ell + g, 0\}$.

The property G_ℓ in Theorem 1.1 is a rather weak requirement on the local number of generators of I: it means that the minimal number of generators $\mu(I_\mathfrak{p})$ is at most

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