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On Elliptic Dunkl Operators

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1. Introduction

Elliptic Dunkl operators for Weyl groups were introduced in [BuFV]. Another version of such operators was considered by Cherednik [C1], who used them to prove the quantum integrability of the elliptic Calogero–Moser systems. The goal of the present paper is to define elliptic Dunkl operators for any finite group W acting on a (compact) complex torus X. (Although we work with a general finite group W, the theory essentially reduces to the case when W is a crystallographic reflection group [GeM, 5.1] because W can be replaced by its subgroup generated by reflections.) We attach such a set of operators to any topologically trivial holomorphic line bundle $\mathcal{L}$ on X with trivial stabilizer in W and to any flat holomorphic connection ∇ on this bundle. When W is the Weyl group of a root system and X is the space of homomorphisms from the root lattice to the elliptic curve, our operators coincide with those of [BuFV]. We prove that the elliptic Dunkl operators commute, and we show that the monodromy of the holonomic system of differential equations defined by them gives rise to a family of $|W|$-dimensional representations of the Hecke algebra $\mathcal{H}_\tau(X, W)$ of the orbifold X/W defined in [E]; conjecturally, this gives generic irreducible representations of this algebra. In the case of Weyl groups, the algebra $\mathcal{H}_\tau(X, W)$ is the double affine Hecke algebra (DAHA) of Cherednik [C2], while in the case $W = S_n \ltimes (\mathbb{Z}/\ell Z)^n$ ($\ell = 2, 3, 4, 6$) it is the generalized DAHA introduced in [EGO]. We reproduce known families of representations of these algebras and also explain how to use the elliptic Dunkl operators to construct representations from category $\mathcal{O}$ over the elliptic Cherednik algebra—that is, the Cherednik algebra of the orbifold X/G defined in [E].

In future work we plan to use the elliptic Dunkl operators to construct new quantum integrable systems. Namely, we expect that for any X and a complex reflection group W acting on X there exists a commuting system of differential operators $L_1, \dots, L_d$, where $d = \dim X$, whose symbols are generators of the ring of W-invariant polynomials on the tangent space to X at the origin. This system is supposed to depend on the same collection of parameters ("coupling constants") as the elliptic Dunkl operators (except for the bundle $\mathcal{L}$, on which it should be independent), and it should be obtained by appropriately symmetrizing the elliptic

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