Michigan Math. J. 57 (2008)

Structure Theorems for Certain Gorenstein Ideals

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Dedicated to Melvin Hochster, on the occasion of his sixty-fifth birthday

1. Introduction

Let I be an ideal in the regular local ring $(R, \mathfrak{n})$ such that $I \subseteq \mathfrak{n}^2$, and let

$$A := R/I, \quad \mathfrak{m} := \mathfrak{n}/I, \quad \mathbf{k} := R/\mathfrak{n} = A/\mathfrak{m}.$$

Let $d = \dim(A)$ be the dimension, e the multiplicity, and $h = v(\mathfrak{m}) - d$ the embedding codimension of A. We assume that $\mathbf{k}$ is a field of characteristic 0 (see the comment after Proposition 2.3).

A classical problem in the theory of local rings is the determination of the minimal number of generators $v(I) := \dim_k(I/\mathfrak{n}I)$ of the ideal I under certain restrictions on the numerical characters of A. For example, by a classical theorem of Abhyankar we know that $e \geq h + 1$, and if the equality $e = h + 1$ holds then we say that A has *minimal multiplicity* and we know that $v(I) = \binom{h+1}{2}$.

In a sequence of papers, Rosales and García-Sánchez proved the following results for A the one-dimensional local domain corresponding to a monomial curve in the affine space (see [4; 5; 6]). By difficult computations related to the numerical semigroup of the curve, they were able to prove the following: if $h + 2 \leq e \leq h + 3$, then

$$\binom{h+2}{2} - e \leq v(I) \leq \binom{h+1}{2}; \tag{1}$$

if $h + 2 \leq e \leq h + 4$ and A is Gorenstein, then

$$v(I) = \binom{h+1}{2} - 1. \tag{2}$$

We remark that the monomial curve $\{t^8 : t^{10} : t^{12} : t^{15}\}$ shows that (2) does not hold if $e = h + 5$ (see [6]).

On the other hand, the monomial curve $\{t^7 : t^8 : t^{10} : t^{19}\}$ shows that the upper bound in (1) does not hold if $e = h + 4$. In the same paper it is asked whether

$$\binom{h+2}{2} - e = \binom{h+1}{2} - 3 \leq v(I) \leq \binom{h+1}{2} + 1 \tag{3}$$

holds for $e = h + 4$.

Received June 20, 2007. Revision received September 13, 2007.

J.E. was partially supported by MTM2007-67493. G.V. was partially supported by the Consiglio Nazionale delle Ricerche (CNR).