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Row Ideals and Fibers of Morphisms

David Eisenbud & Bernd Ulrich

Affectionately dedicated to Mel Hochster,
who has been an inspiration to us for many years,
on the occasion of his 65th birthday

1. Introduction

In this paper we study the fibers of a rational map from an algebraic point of view. We begin by describing four ideals related to such a fiber.

Let $S = k[x_0, \dots, x_n]$ be a polynomial ring over an infinite field k with homogeneous maximal ideal $\mathbf{m}$, let $I \subset S$ be an ideal generated by an $(r+1)$-dimensional vector space W of forms of the same degree, and let ϕ be the associated rational map $\mathbf{P}^n \to \mathbf{P}^r = \mathbf{P}(W)$. We will use this notation throughout. Since we are interested in the rational map, we may remove common divisors of W and thus assume that I has codimension ≥ 2.

A k-rational point q in the target $\mathbf{P}^r = \mathbf{P}(W)$ is by definition a codimension 1 subspace W_q of W. We write $I_q \subset S$ for the ideal generated by W_q. By a homogeneous presentation of I we will always mean a homogeneous free presentation of I with respect to a homogeneous minimal generating set. If $F \to G = S \otimes W$ is such a presentation, then the composition $F \to G \to S \otimes (W/W_q)$ is called the *generalized row* corresponding to q, and its image is called the *generalized row ideal* corresponding to q. It is the ideal generated by the entries of a row in the homogeneous presentation matrix after a change of basis. From this we see that the generalized row ideal corresponding to q is simply $I_q : I$.

The rational map ϕ is a morphism away from the algebraic set $V(I)$, and we may form the fiber (= preimage) of the morphism over a point $q \in \mathbf{P}^r$. The saturated ideal of the scheme-theoretic closure of this fiber is $I_q : I^\infty$, which we call the *morphism fiber ideal* associated to q.

The rational map ϕ gives rise to a *correspondence* $\Gamma \subset \mathbf{P}^n \times \mathbf{P}^r$, which is the closure of the graph of the morphism induced by ϕ. There are projections

$$\mathbf{P}^n \xleftarrow{\pi_1} \Gamma \xrightarrow{\pi_2} \mathbf{P}^r,$$

and we define the *correspondence fiber* over q to be $\pi_1(\pi_2^{-1}(q))$. Since Γ is $\operatorname{BiProj}(\mathcal{R})$, where $\mathcal{R}$ is the Rees algebra $S[It] \subset S[t]$ of I, it follows that the correspondence fiber is defined by the ideal

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