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Global Division of Cohomology Classes via Injectivity

Lawrence Ein & Mihnea Popa

Dedicated to Mel Hochster on his 65th birthday

1. Introduction

The aim of this note is to remark that the injectivity theorems of Kollár and Esnault–Viehweg can be used to give a quick algebraic proof of a strengthening (by dropping the positivity hypothesis) of the Skoda-type division theorem for global sections of adjoint line bundles vanishing along suitable multiplier ideal sheaves (proved in [EL]) and to extend this result to higher cohomology classes as well (cf. Theorem 4.1). For global sections, this is a slightly more general statement of the algebraic version of an analytic result of Siu [S] based on the original Skoda theorem. In Section 4 we list a few consequences of this type of result, such as the surjectivity of various multiplication or cup product maps and the corresponding version of the geometric effective Nullstellensatz.

Along the way, in Section 3 we write down an injectivity statement for multiplier ideal sheaves (Theorem 3.1) and its implicit torsion-freeness and vanishing consequences (Theorem 3.2). These statements are not required in this generality for the main result here (see the following paragraph), but having them available will hopefully be of use. Modulo some standard tricks, the results in Section 3 reduce quickly to theorems of Kollár [K1] and Esnault and Viehweg [EsV], and we do not claim originality in any of the proofs.

All of the results are proved in the general setting of twists by nef and abundant (or good) line bundles, which replace twists by nef and big line bundles required for the use of vanishing theorems. In particular, what is used in the proof of the main Skoda-type statement is a Kollár vanishing theorem for the higher direct images of adjoint line bundles of the form $K_X + L$, where L is the round-up of a nef and abundant $\mathbf{Q}$-divisor. For such vanishing, the only contribution we bring here is a natural statement that seems to be slightly more general that what we found in the literature (cf. Corollary 3.3(4)). The proof is otherwise standard after establishing a simple lemma on restrictions of nef and abundant divisors in Section 2.

Mel Hochster used tight closure techniques to give a beautiful treatment to local statements of Briançon–Skoda type in positive characteristic. We are very happy to be able to contribute work in a similar circle of ideas to a volume in his honor.

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