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Intersection Multiplicities, the Canonical Element Conjecture, and the Syzygy Problem

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In honor of Melvin Hochster on the occasion of his 65th birthday

Introduction

In this paper we shall concentrate on the canonical element conjecture due to M. Hochster as well as several of its ramifications. In [17] Hochster introduced a number of equivalent forms of this conjecture and proved it in the equicharacteristic case. One of the earliest forms, the direct summand conjecture, was proved by Hochster [15] a decade earlier under the same hypothesis (see also [16]). In 1981, Evans and Griffith [10] gave an affirmative answer to the syzygy problem for equicharacteristic local rings. In the course of their proof, they implicitly established a result for finite free complexes [10] that Hochster explicitly isolated in his article [17]. He referred to the new result as the "improved new intersection theorem" (henceforth INIT) because it is clear that INIT implies the new intersection theorem [17]. Of course, INIT remains a conjecture in the case of mixed characteristic. In the same article [17] Hochster pointed out that INIT was a consequence of the canonical element conjecture, and later the first author [3] showed that the two conjectures are equivalent. Over the years, several special cases of the canonical element conjecture have been proved and new equivalent forms have been introduced (see [2; 3; 5; 6; 7; 8; 14]). The four main equivalent versions of this conjecture, that is, the direct summand conjecture, the monomial conjecture, the canonical element conjecture, and the improved new intersection conjecture along with the statement of the syzygy problem are given at the end of this section.

In Section 1 our main focus will be on a formulation of the monomial conjecture in terms of comparing the lengths "Tor_0" and "Tor_1" of a pair of finitely generated modules over a regular local ring. Given a local ring A and a pair of finitely generated modules M and N such that $\ell(M \otimes_A N) < \infty$ (where ℓ denotes length), we raise the following question:

$$\text{Is } \ell(M \otimes_A N) > \ell(\mathrm{Tor}_1^A(M, N))? \qquad \text{(Q)}$$

It is clear that (Q) has obvious negative answers—for example, when $M = N = K$, the residue field of A, or when $M = K$ and $N = m$, the maximal ideal of A. To get to the heart of the matter for A, a regular local ring, we first observe that (Q) boils down to the following question:

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