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Hyperplane Arrangements and Box Splines

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(with an Appendix by A. Björner)

Dedicated to Mel Hochster on the occasion of his 65th birthday

1. Introduction

The purpose of this paper is to explain some explicit formulas that one can develop in the theory of the box spline and the corresponding algorithms of approximation by functions—in particular, (2.11) and (3.2). The theory of splines is a large subject, and even the part on the box spline is rather well developed. The reader should consult the fundamental book by de Boor, Höllig, and Riemenschneider [7] or the recent notes by Ron [19]. Of this large theory we concentrate on some remarkable theorems of Dahmen and Micchelli (see [3; 4; 5]) and on the theory of quasi-interpolants and the Strang–Fix conditions, for which we refer to de Boor [6].

In essence, here we make explicit certain constructs that are already present in [6]. Thus, from a purely computational point of view, there is probably no real difference with that paper, yet we believe that the explicit formulas (2.11) and (3.2) shed a light on the whole procedure. In fact, the main new formula is (3.2) since (2.11) is essentially in Dahmen–Micchelli (although not so explicit).

We also show how some facts about matroids, which are recalled in an appendix written by A. Björner, give a proof of one of the basic theorems of the theory on the dimension of a certain space of polynomials describing box splines locally.

2. Preliminaries

2.1. Box Splines

The theory has been developed in the general framework of approximation theory by splines—in particular, two special classes of functions: the *multivariate spline* $T_X(x)$ and the *box spline* $B_X(x)$.

Take a finite list $X := \{a_1, \dots, a_m\}$ of nonzero vectors $a_i \in V = \mathbb{R}^s$, thought of as the columns of a matrix A. If X spans $\mathbb{R}^s$, one builds an important function for numerical analysis, the *box spline* $B_X(x)$, which is implicitly defined by the formula

$$\int_{\mathbb{R}^s} f(x) B_X(x)\, dx := \int_0^1 \cdots \int_0^1 f\Big(\sum_{i=1}^m t_i a_i\Big)\, dt_1 \dots dt_m, \tag{2.1}$$

where $f(x)$ varies in a suitable set of test functions.

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