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On Injectivity of Maps between Grothendieck Groups Induced by Completion

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Dedicated to Professor Melvin Hochster on the occasion of his sixty-fifth birthday

1. Introduction

Let (R, m, k) be a local ring and let $\hat{R}$ be the m-adic completion of R. Let $\mathcal{M}(R)$ be the category of finitely generated R-modules. The Grothendieck group of finitely generated modules over R is defined as

$$\mathrm{G}(R) = \frac{\bigoplus_{M \in \mathcal{M}(R)} \mathbb{Z}[M]}{\langle [M_2] - [M_1] - [M_3] \mid 0 \to M_1 \to M_2 \to M_3 \to 0 \text{ is exact} \rangle}.$$

Kamoi and Kurano [KKu] studied injectivity of the map $\mathrm{G}(R) \to \mathrm{G}(\hat{R})$ induced by flat base-change. They showed that such a map is injective in the following cases: (i) R is Hensenlian, (ii) R is the localization at the irrelevant ideal of a positively graded ring over a field, or (iii) R has only isolated singularity. Their results raise the question: Is the map between Grothendieck groups that is induced by completion always injective?

In [H1] Hochster announced a counterexample to this question as follows.

Theorem 1.1. *Let k be a field, and let*

$$R = k[x_1, x_2, y_1, y_2]_{(x_1, x_2, y_1, y_2)}/(x_1x_2 - y_1x_1^2 - y_2x_2^2).$$

Let $P = (x_1, x_2)$ and $M = R/P$. Then $[M]$ is in the kernel of the map $\mathrm{G}(R) \to \mathrm{G}(\hat{R})$. However, $[M] \neq 0$ in $\mathrm{G}(R)$.

Hochster's example comes from the "direct summand hypersurface" in dimension 2 and is not normal. He claimed that there should also be an example that is normal. The main purpose of this note is to provide such an example by proving the following result.

Proposition 1.2. *Let $R = \mathbb{R}[x, y, z, w]_{(x,y,z,w)}/(x^2 + y^2 - (w+1)z^2)$, where R is a normal domain. Let $P = (x, y, z)$ and $M = R/P$. Then $[M]$ is in the kernel of the map $\mathrm{G}(R) \to \mathrm{G}(\hat{R})$. However, $[M] \neq 0$ in $\mathrm{G}(R)$.*

This will be proved in Section 2. We note that Kurano and Srinivas [KuS] have recently constructed an example of a local ring R such that the map $\mathrm{G}(R)_{\mathbb{Q}} \to$

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