Semigroups of Valuations on Local Rings

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Dedicated to Mel Hochster on the occasion of his sixty-fifth birthday

Introduction

In recent years the progress and applications of valuation theory have brought to light the importance of understanding the semigroups of values that a Krull valuation ν of some field takes on a Noetherian local ring (R, m) contained in the ring (R_ν, m_ν) of the valuation.

Two general facts about valuations dominating a Noetherian local domain are proven by Zariski and Samuel in their book on commutative algebra [17]. The first is that these semigroups are well-ordered subsets of the positive part of the value group and of ordinal type at most ω^h, where ω is the ordinal type of the well-ordered set $\mathbf{N}$ and h is the rank of the valuation (see [17, Apx. 3, Prop. 2]). Being well-ordered, each value semigroup of a Noetherian ring has a unique minimal system of generators that is indexed by an ordinal no greater than ω^h.

The second general fact is that, if $m_\nu \cap R = m$ and if R and R_ν have the same field of fractions, then the Abhyankar inequality

$$\mathrm{rr}(\nu) + \mathrm{tr}_k k_\nu \leq \dim R$$

holds for the rational rank of the group of the valuation, the transcendence degree of the residue field of R_ν as an extension of the residue field of R, and the dimension of R. If equality holds then the group of the valuation is isomorphic to $\mathbf{Z}^{\mathrm{rr}(\nu)}$ (see [17, Apx. 2]). If, in addition, the rank of h is 1, then the ν-adic and m-adic topologies of R coincide (see [14, Prop. 5-1]).

These two conditions on ordinal type and rational rank do not characterize value semigroups of Noetherian rings. In [5], a third condition is given that concerns the rate of growth of the number of generators of the semigroup of a valuation dominating an equicharacteristic Noetherian local domain. This condition implies that there exist subsemigroups of $\mathbf{Q}_+$ that are well-ordered of ordinal type ω—and so satisfy the first two conditions—but are not semigroups of valuations dominating an equicharacteristic Noetherian local domain.

Examples, starting with plane branches (see [15; 16]) and continuing with quasi-ordinary hypersurfaces (see [8]) suggest that the structure of the semigroup contains important information on the process of local uniformization.