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Canonical Hilbert–Burch Matrices for Ideals of $k[x, y]$

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1. Introduction

Let k be a field of arbitrary characteristic and R the polynomial ring $k[x_1, \dots, x_n]$. Let τ be a term order on R. Given a nonzero $f \in R$, we denote by $\mathrm{Lt}_\tau(f)$ the largest term with respect to τ appearing in f and by $\mathrm{Lc}_\tau(f)$ the coefficient of $\mathrm{Lt}_\tau(f)$ in f. For an ideal I of R we denote by $\mathrm{Lt}_\tau(I)$ the (monomial) ideal generated by $\mathrm{Lt}_\tau(f)$ with $f \in I \setminus \{0\}$. Let E be a monomial ideal of R. Consider the set $V(E)$ of the homogeneous ideals I of R such that $\mathrm{Lt}_\tau(I) = E$. The set $V(E)$ has a natural structure of affine variety. Namely, given I in $V(E)$, we can consider I as a point in an affine space $\mathbf{A}^N$ with coordinates given by the coefficients of the nonleading terms in the reduced Gröbner basis of I; see Section 2 for details. The equations defining (at least set-theoretically) $V(E)$ can be obtained from Buchberger's Gröbner basis criterion. Provided $\dim_k R/E$ is finite, one can give the structure of affine variety also to the set $V_0(E)$ of the ideals I (homogeneous or not) such that $\mathrm{Lt}_\tau(I) = E$.

These and similarly defined varieties play important roles in many contexts, such as the study of various types of Hilbert schemes and the problem of deforming nonradical to radical or prime ideals (see [ASt; Br; CRV; ES1; ES2; G1; G2; Ha; H1; H2; I1; I2; IY; KRo2; MSt]).

Many of the equations defining $V(E)$ or $V_0(E)$ contain parameters that appear in degree 1 and that can be eliminated. It happens quite often that, after dispensing with the superfluous parameters, one is left with no equations—that is, the variety is an affine space. But it is well known that, in general, $V(E)$ can be reducible and can have irreducible components that are not affine spaces; see Examples 2.1–2.3.

On the other hand, for $n = 2$ and $d = \dim_k R/E < \infty$ it is known that $V_0(E)$ and $V(E)$ are affine spaces. This has been proved (for some term orders) by Briançon [Br] and Iarrobino [I1]; for general τ, it follows from a general result of Bialynicki-Birula [Bi1; Bi2] on smooth varieties with k^*-actions. Here it is important to note that $V_0(E)$ coincides with the set of points of the Hilbert scheme $\mathrm{Hilb}^d(\mathbf{A}^2)$ that degenerate to E under a suitable k^*-action associated to a weight vector representing the term order on monomials of degree $\le d+1$. By the analogy with Schubert cells for Grassmannians, we call $V_0(E)$ and $V(E)$ *Gröbner cells*.

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