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Complete Intersections on General Hypersurfaces

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1. Introduction

Many problems in classical projective geometry ask about the nature of special subvarieties of some given family of varieties. For example: How many isolated singular points can a surface of degree d in $\mathbb{P}^3$ have? When is it true that the members of a certain family of varieties contain a line or contain a linear space of any positive dimension? The reader can easily supply other examples of such questions.

This is the kind of problem we consider in this paper: What types of complete intersection varieties of codimension r in $\mathbb{P}^n$ can one find on the generic hypersurface of degree d?

In case $r = 2$ it was known to Severi [Se] that, for $n \geq 4$, the only complete intersections on a general hypersurface are obtained by intersecting that hypersurface with another.

This observation was extended to $\mathbb{P}^3$ by Noether (and Lefschetz) [Le; GrH] for general hypersurfaces of degree ≥ 4. These ideas were further generalized by Grothendieck [Gro].

Our approach to the question just posed uses a mix of projective geometry and commutative algebra and is much more elementary and accessible than, for example, the approach of Grothendieck. We are able to give a complete answer to the question we raised for complete intersections of codimension r in $\mathbb{P}^n$ that lie on a general hypersurface of degree d whenever $2r \leq n+2$. In particular, we treat the case of complete intersections of small codimension. The case of complete intersection curves on hypersurfaces (i.e., complete intersection of small dimension) was treated and solved by Szabó in [Sz].

The paper is organized as follows. In Section 2 we lay out the question we want to consider and explain what are the interesting parameters for a response.

In Section 3 we collect some necessary technical information about varieties of reducible forms and their joins. In order to find the dimensions of these joins (using Terracini's lemma), we calculate the tangent space at a point of any variety of reducible forms. We also recall some information about Artinian complete intersection quotients of a polynomial ring.

In Section 4, we use the technical facts collected in Section 3 to reformulate our original question. We illustrate the utility of this reformulation to discuss complete

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