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Quiver Coefficients of Dynkin Type

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1. Introduction

Let $Q = (Q_0, Q_1)$ be a quiver consisting of a finite set of vertices Q_0 and a finite set of arrows Q_1. Each arrow $a \in Q_1$ has a head $h(a)$ and a tail $t(a)$ in Q_0. For convenience we will assume that the vertex set is an integer interval, $Q_0 = \{1, 2, \dots, n\}$. Let $e = (e_1, \dots, e_n) \in \mathbb{N}^n$ be a dimension vector, and fix vector spaces $E_i = \mathbb{K}^{e_i}$ for $i \in Q_0$ over a field $\mathbb{K}$. The representations of Q on these vector spaces form the affine space $V = \bigoplus_{a \in Q_1} \mathrm{Hom}(E_{t(a)}, E_{h(a)})$, which has a natural action of the group $\mathbb{G} = \mathrm{GL}(E_1) \times \cdots \times \mathrm{GL}(E_n)$ given by $(g_1, \dots, g_n).(\phi_a)_{a \in Q_1} = (g_{h(a)} \phi_a g_{t(a)}^{-1})_{a \in Q_1}$.

Define a *quiver cycle* to be any $\mathbb{G}$-stable closed irreducible subvariety Ω in V. A quiver cycle determines an equivariant (Chow) cohomology class $[\Omega] \in H^*_{\mathbb{G}}(V)$ and an equivariant Grothendieck class $[\mathcal{O}_\Omega] \in K_{\mathbb{G}}(V)$. These classes are well understood when the quiver Q is *equioriented* of type A, that is, a sequence $\{1 \to 2 \to \cdots \to n\}$ of arrows in the same direction. In this case, a formula for the cohomology class $[\Omega]$ was given in joint work with Fulton [11], and this formula was generalized to K-theory in [8]. The K-theory formula states that the Grothendieck class $[\mathcal{O}_\Omega]$ is given by

$$[\mathcal{O}_\Omega] = \sum_\mu c_\mu(\Omega) \mathcal{G}_{\mu_1}(E_2 - E_1) \mathcal{G}_{\mu_2}(E_3 - E_2) \cdots \mathcal{G}_{\mu_{n-1}}(E_n - E_{n-1}) \in K_{\mathbb{G}}(V),$$

where the sum is over finitely many sequences $\mu = (\mu_1, \dots, \mu_{n-1})$ of partitions μ_i. Each factor $\mathcal{G}_{\mu_i}(E_{i+1} - E_i)$ is obtained by applying the stable Grothendieck polynomial for μ_i to the standard representations of $\mathbb{G}$ on E_{i+1} and E_i. This notation will be explained in Section 3.

The coefficients $c_\mu(\Omega)$ are interesting geometric and combinatorial invariants called (equioriented) *quiver coefficients.* They are integers and are nonzero only when the sum $\sum |\mu_i|$ of the weights of the partitions is greater than or equal to the codimension of Ω. The coefficients for which this sum equals $\mathrm{codim}(\Omega)$ describe the cohomology class of Ω and are called *cohomological quiver coefficients.* It was proved in [25] that cohomological quiver coefficients are nonnegative and in [10; 29] that the more general K-theoretic quiver coefficients have *alternating signs* in the sense that $(-1)^{\sum |\mu_i| - \mathrm{codim}(\Omega)} c_\mu(\Omega)$ is a nonnegative integer. These

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