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Gröbner Bases and Betti Numbers of Monoidal Complexes

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To Mel Hochster on his 65th birthday

1. Introduction

Combinatorial commutative algebra is a branch of combinatorics, discrete geometry, and commutative algebra. On the one hand, problems from combinatorics or discrete geometry are studied using techniques from commutative algebra; on the other hand, questions in combinatorics motivated various results in commutative algebra. Since the fundamental papers of Stanley (see [13] for the results) and Hochster [8; 9], combinatorial commutative algebra has been a growing and active field of research. See also Bruns and Herzog [7], Villarreal [16], Miller and Sturmfels [11], and Sturmfels [15] for classical and recent results and new developments in this area of mathematics.

Stanley–Reisner rings and affine monoid algebras are two of the classes of rings considered in combinatorial commutative algebra. In this paper we consider toric face rings associated to monoidal complexes. They generalize Stanley–Reisner rings by allowing a more general incidence structure than simplicial complexes and more general rings associated with their faces—namely, affine monoid algebras instead of polynomial rings.

In cooperation with M. Brun and B. Ichim, the authors have studied the local cohomology of toric face rings in previous work [1; 3; 10]. One of the main results is a general version of Hochster's formula for the local cohomology of a Stanley–Reisner ring (see [7] or [13]) even beyond toric face rings.

In this paper we want to generalize Hochster's formulas for the graded Betti numbers of a Stanley–Reisner ring [9] and affine monoid rings [11, Thm. 9.2] to toric face rings. Such a generalization is indeed possible for monoidal complexes that, roughly speaking, can be embedded into a space $\mathbb{Q}^d$ (see Theorem 4.5). As counterexamples show, full generality does not seem possible. One of the problems encountered is to construct a suitable grading. This forces us to consider grading monoids that are not necessarily cancellative.

Another topic treated in Section 3 is initial ideals (of the defining ideals) of toric face rings with respect to monomial (pre)orders defined by weights. Indeed, toric face rings come up naturally in the study of initial ideals of affine monoid algebras. In this regard we generalize results of Sturmfels [15]. We will pay special attention to the question of when the initial ideal is radical, monomial, or both

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