Michigan Math. J. 57 (2008)

A Remark on Frobenius Descent for Vector Bundles

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Dedicated to Mel Hochster on the occasion of his 65th birthday

1. Introduction

Let X be a smooth projective variety defined over an algebraically closed field of characteristic $p > 0$ with a fixed very ample line bundle $\mathcal{O}_X(1)$. We denote by F the absolute Frobenius morphism $F: X \to X$, which is the identity on the topological space underlying X and the pth power map on the structure sheaf $\mathcal{O}_X$. A vector bundle $\mathcal{E}$ on X descends under F if there exists a vector bundle $\mathcal{F}$ such that $\mathcal{E} \cong F^*(\mathcal{F})$. This paper is inspired by the preprint of Joshi [6]. In the relative situation, where a morphism $\mathcal{X} \to \operatorname{Spec} R$ with generic fiber $X := \mathcal{X}_0$ is given and R is a $\mathbb{Z}$-domain of finite type, Joshi asked the following question: Assume X is a smooth projective variety and suppose V is a vector bundle that descends under Frobenius modulo an infinite set of primes; then is it true that V is semistable (with respect to any ample line bundle on X)?" He gives a positive answer to this question for rank-2 vector bundles under the additional assumption that $\operatorname{Pic}(X) = \mathbb{Z}$.

In Section 2 we provide a class of examples that give a negative answer to this question in general. We show that, on the relative Fermat curve

$$C = V_+(X^d + Y^d + Z^d) \to \operatorname{Spec} \mathbb{Z}$$

with $d \geq 5$ odd, there exists a vector bundle $\mathcal{E}$ of rank 2 such that for infinitely many prime numbers p the reduction $\mathcal{E}_p = \mathcal{E}|_{C_p}$ modulo p has a Frobenius descent, but $\mathcal{E}_0 = \mathcal{E}|_{C_0}$ is not semistable on the fiber over the generic point. In Section 3 we give an affirmative answer to this question under the assumption that, for every closed point $\mathfrak{m} \in \operatorname{Spec} R$, every semistable vector bundle on the fiber $\mathcal{X}_{\mathfrak{m}}$ is strongly semistable. We recall that a semistable vector bundle $\mathcal{E}$ is strongly semistable if $F^{e*}(\mathcal{E})$ is semistable for all integers $e \geq 0$. This provides further examples of varieties with $\operatorname{Pic}(X) \neq \mathbb{Z}$ (e.g., abelian varieties) for which the question of Joshi still has a positive answer.

Acknowledgments. We would like to thank A. Werner for pointing out this problem to us. We also thank the referee for many useful comments that helped to simplify the proof of Lemma 2.1 and to clarify Example 2.5 via Lemma 2.4.

Received September 11, 2007. Revision received February 11, 2008.