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Singularities of Moduli Spaces of Vector Bundles over Curves in Characteristic 0 and p

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Melvin Hochster has made enormous contributions to commutative algebra, not only through his own work but also through the influence he has had on his students, colleagues, and co-authors; this paper is dedicated to him on the occasion of his 65th birthday

1. Introduction

Let X be a nonsingular projective curve over an algebraically closed field k of any characteristic. One has J, the *Jacobian* of X. This parameterizes the isomorphism classes of line bundles of degree 0 on X. This was constructed classically, so now one can use geometric invariant theory. Similiarly, one also considers J^d, the Jacobian of line bundles of degree d, for any integer d. If L is a line bundle of degree d on X, then $M \to M \otimes L$ is an isomorphism from J to J^d; hence J and J^d are isomorphic, but not canonically.

Suppose that k is the field $\mathbb{C}$. Viewing X as a complex manifold, we may consider $\operatorname{Hom}(\pi_1(X), S^1)$. This is isomorphic to $(S^1)^{2g}$, which is also a complex torus of complex dimension g and isomorphic to J. For bundles of rank > 1 (over any field), put $\mu(V) = \deg V/\operatorname{rank} V$. Then we say that V is *stable* (resp. *semistable*) if, for all proper subbundles W of V,

$$\mu(W) < \mu(V) \quad (\text{resp. } \mu(W) \le \mu(V)).$$

Over $\mathbb{C}$, if degree $V = 0$ then one has the following classical result.

Theorem 1.1. *V is stable (resp., a direct sum of stable bundles of degree 0) if and only if $V \simeq V_\sigma$ for an irreducible (arbitrary) $\sigma : \pi_1(X) \to U(r)$, where $U(r)$ is the unitary group on r variables, $r = \operatorname{rank} V$. This σ is unique up to conjugation.*

Now let k be arbitrary, and let $S = (V)$ be the set of semistable bundles of rank r and degree 0 on X. The set S is bounded; that is, there exists an $m \gg 0$ such that, for all V in S, one has $H^1(V(m)) = 0$ and that $H^0(V(m))$ generates $V(m)$. Let $n = H^0(V(m))$ and $G = \mathrm{GL}(n)$. Let H be the Hilbert scheme of quotients of $\mathcal{O}_X^n$,

$$0 \to K \to \mathcal{O}_X^n \to F \to 0.$$

Let R^s (resp. R^{ss}) be the open, G-invariant subset of H consisting of all those quotients F of $\mathcal{O}_X^n$ such that:

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